SOLUTIONS OF THE GENERAL $n$-TH ORDER VARIABLE COEFFICIENTS
LINEAR DIFFERENCE EQUATION

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Abstract

In this paper, variable operator and its product with shifting operator are studied. The product of power series of shifting operator with variable coefficient is defined and its convergence is proved under Mikusiński's sequence convergence. After turning a general variable coefficient linear difference equation of order $n$ into a set of operator equations, we can obtain the solutions of the general $n$-th order variable coefficient linear difference equation.

Key words Mikusiński's operator, variable operator, convergence, linear difference equation with variable coefficients, solution of series form, operator equation

I. Introduction

It is well-known that the famous Polish mathematician J. Mikusiński set up the theory of Operational Calculus[1]. This algebraic system is particularly helpful in solving constant coefficient linear differential or difference equations, and it is better than the method of Laplace transformations. In addition, since we have the sequence convergence, it is possible to use the 1-1 correspondence relationship between the shifting operator series and some classical set of analytic functions to find the solutions of general constant coefficient linear difference equations. In this paper, the product relationship of variable operator and shifting operator is given with the help of variable operator, we define the product of variable coefficient shifting operator series. After turning a general variable coefficient linear difference equation of order $n$ into a set of operator equations, we can acquire the solutions of general $n$-th order variable coefficient linear difference equation.

II. Fundamental Theory

We denote $\mathcal{F}$ the set of all continuous complex valued functions $f=\{f(t)\}$ of a real variable number with the property that for every $f \in \mathcal{F}$ there is a real number $\sigma$ such that $f(t)=0$ for all $t<\sigma$, then $\mathcal{F}$ forms a commutative algebra without zero divisors by Titchmarsh's theorem, where the product is defined as the convolution, that is

$$f \cdot g = \left\{ \sum_{\tau=-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \right\} \quad (f, g \in \mathcal{F})$$

and the sums and the scalar products are defined in the usual way, quotient field $Q$ of this.
algebra which is Mikusiński operator field. There are integral operator \( I = \{ g(t) \} \), differential operator \( s = 1 / I \), shifting operator \( h^\lambda = s \{ H_\lambda (t) \} \) (\( \lambda > 0 \)) etc. in \( Q \), where

\[
g(t) = \begin{cases} 1 & (t \geq 0) \\ 0 & (t < 0) \end{cases}, \quad H_\lambda (0) = \begin{cases} 1 & (t \geq \lambda) \\ 0 & (t < \lambda) \end{cases}
\]

and for every \( \lambda > 0 \), \( f = \{ f(t) \} \in \mathcal{S} \), we have the following conclusion:

(i) \( h^\lambda \{ f(t) \} = \{ f(t - \lambda) \} \)

(ii) \( h^{-\lambda} \{ f(t) \} = \{ f(t + \lambda) \} \)

and \( h^0 = 1, \ h^{-\lambda} = 1 / h^\lambda \).

For every complex-number \( a \in \mathbb{C} \), the number operator \( [a] \) is defined as

\[
[a] = \{ a \} / I
\]

where the function \( \{ a \} \) is zero for all \( t < 0 \) and is \( a \) for all \( t \geq 0 \).

Now for every function of \( \omega = \{ \omega(t) \} \in \mathcal{S} \), \( A(\omega) \geq 0 \) and every \( t \) of \( t \geq 0 \), the variable operator is defined as

\[
\omega(t) = \frac{\{ \omega(t) \}}{I}
\]

where \( A(\omega) = \sup \{ \sigma; \ \omega(t) = 0, \ t < \sigma \} \), \( (\omega \in \mathcal{S}) \).

Therefore, for every function \( \omega = \{ \omega(t) \} \in \mathcal{S}, \ A(\omega) \geq 0 \) which is corresponded to a set of numerical operator \( \{ \omega(t) \} / I \) and it is easy to prove that this correspondence is one-to-one correspondence. (it follows \( \omega(t) + v(t) \overset{1-1}{\leftrightarrow} \{ \omega(t) + v(t) \}, \ \omega(t) v(t) \overset{1-1}{\leftrightarrow} \frac{\{ \omega(t) v(t) \}}{I} \).

We denote by

\[
\mathcal{S}_t = \left\{ \omega(t) = \frac{\{ \omega(t) \}}{I}; \ \omega = \{ \omega(t) \} \in \mathcal{S}, \ A(\omega) \geq 0 \right\}
\]

in order to distinguish, and for every \( a(t) \in \mathcal{S}_t \), \( x = \{ x(t) \} \in \mathcal{S} \), the scalar product is defined as

\[
a(t) \{ x(t) \} = \{ a(t) x(t) \} = \{ x(t) \} a(t),
\]

For every \( x = \{ x(t) \} \in \mathcal{S} \), we get

\[
(a(t) h^\lambda) x = a(t) h^\lambda \{ x(t) \}
\]
\[
(h^\lambda a(t)) x = h^\lambda \{ a(t) \{ x(t) \} \} = h^\lambda \{ a(t) x(t) \}
\]
\[
= \{ a(t - \lambda) x(t - \lambda) \} = a(t - \lambda) h^\lambda \{ x(t) \},
\]

where \( a(t) \in \mathcal{S}_t \), then there is the operation defined as follows

\[
h^\lambda a(t) = a(t - \lambda) h^\lambda \quad \text{or} \quad a(t) h^\lambda = h^\lambda a(t + \lambda)
\]

and

\[
(a(t) h^\lambda)^2 = (a(t) h^\lambda)(a(t) h^\lambda)
\]
\[
= a(t) a(t - \lambda) h^{2\lambda}
\]