A METHOD OF DETERMINING BUCKLED STATES OF THIN PLATES AT
A DOUBLE EIGENVALUE*

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Abstract
A method of determining bifurcation directions at a double eigenvalue is presented by
combining the finite element method with the perturbation method. By using the present
method, the buckled states of rectangular plates at a double eigenvalue are numerically
analyzed. The results show that this method is effective.

Key words double eigenvalue, bifurcation directions, FEM, perturbation
method, buckling of rectangular plates

I. Introduction
The buckling and post-buckling of plates and shells is one of the important problems in
engineering and technology. Therefore, it is necessary to analyze buckling and buckled states. So
far, the main methods for obtaining the buckled states are approximate and numerical methods, for
example, the perturbation method[10], the finite difference method[11] and the finite element
method[12-5], etc. For the buckling of structures, there are some effective techniques[6] to compute the
buckled states at simple eigenvalues; however, at multiple eigenvalues, there are only a few methods
to compute bifurcation solutions in which the difficulty is the determination of branching
directions. For the buckling of thin plates, it is not effective to use the algebraic bifurcation
equations[6,7] since the nonlinear term of the operator equation is homogeneous of degree 3[8,9]. If we
use the method in [10], the numerical computation is very difficult. In the present paper, on the basis
of the generalized variational principles and the finite element method established in [4, 5, 9], a
discrete system of equations is obtained and a technique for determining directions at a double
eigenvalue is suggested by using the perturbation method. As examples, the buckled states of
rectangular thin plates with both the clamped and simply-supported edge conditions at a double
eigenvalue are analyzed and the eight bifurcation solutions (buckled states) are obtained by the
continuation calculation method.

II. Discrete Equations of the Buckling of Thin Plates
Following [4, 5], the discrete equations in terms of dimensionless variables, using the finite
element method, can be written as follows:

\[ F(x, \lambda) = (A - \lambda C)x + F',(x)x = 0 \]  \hspace{1cm} (2.1)

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in which $x$ is a vector of generalized nodal point parameters, $\lambda$ is a dimensionless load parameter, and $A$, $C$, and $F_i(x)$ are all $n \times n$ real matrices, where $n$ is the number of generalized nodal point parameters and $F_i(x)$ has the property described as follows:

$$F_1(\alpha x^{(1)} + \beta x^{(2)}) = \alpha F_1(x^{(1)}) + \beta F_1(x^{(2)})$$

(2.2)

Obviously, for any $\lambda \in \mathbb{R}^1$, $x \equiv 0$ is one solution of (2.1) and it corresponds to the unbuckled state of thin plates. Along the path of the unbuckled state, seeking singular points is equivalent to seeking eigenvalues of the generalized eigenvalue problem

$$(A - \lambda C)x = 0$$

(2.3)

we can use the subspace iteration method to compute the eigenvalues and the corresponding eigenvectors of (2.3).

III. Determination of Branching Directions

Assume that $\lambda^*$ is an eigenvalue of (2.3) and that $\dim \text{Null}(F_x(0, \lambda^*)) = 2$, that is, $\lambda^*$ is a double eigenvalue. Let $\text{Null}(F_x(0, \lambda^*)) = \text{span}\{e_1, e_2\}; \text{ here, } \langle Ce_i, e_j \rangle = \delta_{ij}, (i, j = 1, 2)$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$. Assume that the bifurcation solutions of (2.1) at $\lambda = \lambda^*$ have the following form:

$$x = e_1(\alpha_1 e_1 + \alpha_2 e_2) + \varepsilon^2 e^{(2)} + \varepsilon^3 x^{(3)} + \cdots$$

$$\lambda = \lambda^* + \varepsilon^{(1)} + \varepsilon^2 \lambda^{(2)} + \cdots$$

(3.1a, b)

where $\varepsilon$ is a small parameter defined by

$$\langle x, (\alpha_1 e_1 + \alpha_2 e_2) \rangle = \varepsilon$$

(3.2)

Both $\alpha_1$ and $\alpha_2$ are undetermined constants.

Substituting (3.1a, b) into (2.1) and comparing the coefficients of like-power on $\varepsilon$, we have

$$(A - \lambda^* C)(\alpha_1 e_1 + \alpha_2 e_2) = 0$$

(3.3a)

$$\alpha_1^2 + \alpha_2^2 = 1$$

(3.3b)

$$(A - \lambda^* C)x^{(2)} - \lambda^{(1)} C(\alpha_1 e_1 + \alpha_2 e_2) + F_i(\alpha_1 e_1 + \alpha_2 e_2)(\alpha_1 e_1 + \alpha_2 e_2) = 0$$

(3.4)

It is not difficult to see that (3.3a) is automatically satisfied. The solvability conditions of (3.4) are

$$\lambda^{(1)} \langle e_1, C(\alpha_1 e_1 + \alpha_2 e_2) \rangle = \langle e_1, F_i(\alpha_1 e_1 + \alpha_2 e_2)(\alpha_1 e_1 + \alpha_2 e_2) \rangle$$

(3.5a)

$$\lambda^{(1)} \langle e_2, C(\alpha_1 e_1 + \alpha_2 e_2) \rangle = \langle e_2, F_i(\alpha_1 e_1 + \alpha_2 e_2)(\alpha_1 e_1 + \alpha_2 e_2) \rangle$$

(3.5b)

Observing $\langle Ce_i, e_j \rangle = \delta_{ij}$, we may obtain the value of $\lambda^{(1)}$ from (3.5) and from the two latter examples in this paper we can get $\langle e_i, F_i(\alpha_1 e_1 + \alpha_2 e_2) \rangle = 0 (i, j, k = 1, 2)$. For the general case, $\langle e_i, F_i(e_j)e_k \rangle = 0$ is also valid (see [4]).

If the right-hand side terms of (3.5a, b) do not vanish, then we get a system of equations determining $(\alpha_1, \alpha_2, \lambda^{(1)})$ from (3.3b) and (3.5). Hence, under the conditions that $\langle x^{(2)}, e_i \rangle = 0 (i = 1, 2), \langle e_i, F_i(e_j)e_k \rangle = 0 (i, j, k = 1, 2)$, we obtain the value of $x^{(3)}$ from (3.4).

If $\langle e_i, F_i(e_j)e_k \rangle = 0 (i, j, k = 1, 2)$, then it is easy to see that $\lambda^{(1)} = 0$. In this case, we have to analyze the term on $\varepsilon^2$ and get

$$(A - \lambda^* C)x^{(2)} - \lambda^{(1)} C(\alpha_1 e_1 + \alpha_2 e_2) + F_i(\alpha_1 e_1 + \alpha_2 e_2)x^{(2)} + F_i(x^{(2)})(\alpha_1 e_1 + \alpha_2 e_2) = 0$$

(3.6)

Assume that $x^{(2)}_{i,j}$ satisfies the following equations

$$(A - \lambda^* C)x^{(2)}_{i,j} = -F_i(e_i)e_j$$

(3.7)

then we have $x^{(2)} = \sum_{i,j=1}^2 a_i a_j x^{(2)}_{i,j}$ when $\langle x^{(2)}_{i,j}, e_k \rangle = 0 (i, j, k = 1, 2)$. Thus, the solvability