GENERALIZED COMPLEMENTARITY PROBLEMS
FOR FUZZY MAPPINGS*

Zhang Shi-sheng (张石生) Huang Nan-jing (黄南京)
(Department of Mathematics, Sichuan University, Chengdu)
(Received July 1, 1991)

Abstract
In this paper we introduce a new class of generalized complementarity problems for the fuzzy mappings and construct a new iterative algorithm. We also discuss the existence of solutions for the generalized complementarity problems and the convergence of iterative sequence.

Key words fuzzy mapping, generalized complementarity problem, algorithm

I. Introduction
The complementarity theory developed by Lemke[1], and Cottle and Dantzig[2] and others in the early 1960s and thereafter, has numerous applications in diverse fields of mathematical and engineering sciences. Recently, complementarity problems have been extended and generalized in various directions to study a large class of problems arising in control and optimization, economics and transportation equilibrium, contact problems in elasticity, and fluid through porous media (see [3 – 5], and the references therein.)

Inspired and motivated by the recent research work in this field, in this paper we introduce a new class of generalized complementarity problems for the fuzzy mappings and construct a new iterative algorithm. We also discuss the existence of solutions for the generalized complementarity problems and the convergence of iterative sequence.

II. Preliminaries and Formulations
Let \( \mathcal{F}(\mathbb{R}^n) \) be a collection of all fuzzy sets over \( \mathbb{R}^n \). A mapping \( F \) from \( \mathbb{R}^n \) into \( \mathcal{F}(\mathbb{R}^n) \) is called fuzzy mapping over \( \mathbb{R}^n \). If \( F \) is a fuzzy mapping over \( \mathbb{R}^n \), then \( F(x) \) (denoted by \( F_x \) in the sequel) is a fuzzy set over \( \mathbb{R}^n \), and \( \mu_{F_x}(y) \) is the membership function of the point \( y \) in \( F_x \).

Let \( B \in \mathcal{F}(\mathbb{R}^n), \ p \in [0,1] \). Then the set
\[
(\mathbb{B})_p = \{ x \in \mathbb{R}^n : B(x) \geq p \}
\]
is called a \( p \)-cut set of \( B \).

Let \( F : \mathbb{R}^n \rightarrow \mathcal{F}(\mathbb{R}^n) \) be a fuzzy mapping which is said to satisfy the condition (I), if it satisfies the following condition:

(I) There exists a function \( p : \mathbb{R}^n \rightarrow [0,1] \) such that for all \( x \in \mathbb{R}^n \), the cut set \( (F_x)_p, x \in C(\mathbb{R}^n) \), where \( C(\mathbb{R}^n) \) denote the family of all nonempty compact subsets of \( \mathbb{R}^n \).

By using the fuzzy mapping \( F \) we can define a set-valued mapping \( \mathcal{F} \) as follows:
In the sequel, we shall call \( \mathcal{F} \) the set-valued mapping induced by fuzzy mapping \( F \).

We denote by \((\cdot , \cdot)\) and \( \| \cdot \| \) the inner product and norm on \( \mathbb{R}^n \) respectively.

Given mappings \( T : \mathbb{R}^n \to \mathbb{R}^n \), \( p : \mathbb{R}^n \to [0,1] \), nonlinear mapping \( A : \mathbb{R}^n \to \mathbb{R}^n \) and fuzzy mapping \( F : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n) \), we consider the problem of finding \( u, y \in \mathbb{R}^n \) such that \( F_u(y) \geqslant p(u) \), and

\[
\begin{align*}
u \geqslant 0, \quad Tu + Ay &\geqslant 0, \quad (u, Tu + Ay) = 0 \\
(2.1)
\end{align*}
\]

If \( T \) is a nonlinear mapping, then problem (2.1) is called generalized strongly nonlinear complementarity problem for fuzzy mapping. If \( T \) is an affine transformation of the form \( u \mapsto Mu + q \), with \( M \in \mathbb{R}^{n \times n} \) and \( q \in \mathbb{R}^n \), then problem (2.1) is equivalent to finding \( u, y \in \mathbb{R}^n \) such that \( F_u(y) \geqslant p(u) \), and

\[
\begin{align*}
u \geqslant 0, \quad Mu + q + Ay &\geqslant 0, \quad (u, Mu + q + Ay) = 0 \\
(2.2)
\end{align*}
\]

which is called the generalized mildly nonlinear complementarity problem for fuzzy mapping.

If \( G : \mathbb{R}^n \to 2^{\mathbb{R}^n} \) is a set-valued mapping. By using \( G \), we can define a fuzzy mapping \( F : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n) \), \( x \mapsto \chi_{G(x)} \), where \( \chi_{G(x)} \) is the characteristic function of the set \( G(x) \). Take \( p(x) \equiv 1 \) for all \( x \in \mathbb{R}^n \). Hence the problems (2.1) and (2.2) are equivalent to find \( u \in \mathbb{R}^n \), \( y \in F(u) \) such that

\[
\begin{align*}
u \geqslant 0, \quad Tu + Ay &\geqslant 0, \quad (u, Tu + Ay) = 0 \\
(2.3)
\end{align*}
\]

and

\[
\begin{align*}
u \geqslant 0, \quad Mu + q + Ay &\geqslant 0, \quad (u, Mu + q + Ay) = 0 \\
(2.4)
\end{align*}
\]

respectively.

If \( G : \mathbb{R}^n \to \mathbb{R}^n \) is an identity mapping, then problem (2.3) is equivalent to finding \( u \in \mathbb{R}^n \) such that

\[
\begin{align*}
u \geqslant 0, \quad Tu + Au &\geqslant 0, \quad (u, Tu + Au) = 0 \\
(2.5)
\end{align*}
\]

which is called strongly nonlinear complementarity problem and problem (2.4) is equivalent to finding \( u \in \mathbb{R}^n \), such that

\[
\begin{align*}
u \geqslant 0, \quad Mu + q + Au &\geqslant 0, \quad (u, Mu + q + Au) = 0 \\
(2.6)
\end{align*}
\]

which is called mildly nonlinear complementarity problem (see [6, 7]).

Problems (2.5) and (2.6) arise, for instance, as finite difference (finite element) approximations to constrained partial differential inequalities of the type

\[
\begin{align*}
-Lu(x) + f(x,u(x)) &\geqslant 0, \quad \text{in } D \quad u(x) \geqslant 0, \quad \text{in } D \\
(2.7)
\end{align*}
\]

where \( L \) is a given nonlinear (linear) elliptic operator, \( D \subset \mathbb{R}^n \) is a domain with boundary \( S \), \( f(u) = f(x,u(x)) \) is a nonlinear function of \( x \) and \( u(x) \), and \( g \) is a given function. Well-known examples of free boundary value problems which can be written in the form (2.7) include fluid flow through porous media, journal bearing lubrication problems and contact problems in elasticity (see [3, 4, 5]).

Obviously, problems (2.2) – (2.6) are all the special cases of problem (2.1).

Problems (2.1) and (2.2) can be written as