SINGULARITY UNDER A CONCENTRATED FORCE IN ELASTICITY*

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Abstract

We first discuss singularity problem of a sort of partial differential equation involving δ function. Using this result we then have the answer to various singularity problems in elasticity due to the presentation of a concentrated force. Lastly corresponding conclusions in vibration problem are drawn.

Key words elasticity, concentrated force, singularity

In elasticity it is important to study the singularity under concentrated forces. Some problems can be solved directly\(^\text{[3-5]}\), such as the elastic solution of half-infinite plane under a concentrated force. Other problems can be solved by simplifying the model, such as the axial symmetrical solution of a circular plate. But it is very complex to analyze the singularity of many questions by solving them directly. Besides, the reasonableness of theories of structures carrying concentrated masses, springs and supports in vibration problems relies on the singularity in static mechanics\(^\text{[1]}\). Thus it is quite useful to give the result of singularity under concentrated force in elasticity. We derive the answer by analyzing the singularity of a sort of partial differential equations involving δ function.

1. Consider an \(m\)-dimensional partial differential equation

\[
D^n u(x) = P(x)
\]

where \(x = (x_1, x_2, \ldots, x_m)\), \(D^n\) is the differential factor, whose expression is

\[
D^n = \sum_{a_1} \sum_{a_2} \cdots \sum_{a_m} \frac{\partial^a}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_m^{a_m}}
\]

\(a_1, a_2, \ldots, a_m\) are non-negative integers and there exists \(\sum_{i=1}^{m} a_i = n\).

For a specific mechanics problem, \(a_1, a_2, \ldots, a_m\) are selected definitely, and there exists the possibility that \(a_i = n\) \((i = 1, 2, \ldots, m)\).

For a concentrated force, in (1) we take

\[
P(x) = F \delta(x - x_0)
\]

where \(x_0\) is the point of application of the force.

Then one integral form of (1) is

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where each \( e_1, e_2, \ldots, e_m \) is small, and for convenience we may set
\[
e_i = e \quad (i = 1, 2, \ldots, m)
\]

Theorem  

a) The solution \( u(x) \) is singular at the point \( x_0 \) if \( n \leq m \) and the singularity order is
\[
O(e^{n-m}), \quad \text{if } n < m
\]
\[
O(\ln e), \quad \text{if } n = m
\]
b) \( u(x) \) is not singular at the point \( x_0 \) if \( n > m \).

Below we give a simple proof.

a) From the equation (2), we know the order of singularity of \( D^n u(x) \) at the point \( x_0 \) is \( O(1/e^{n}) \). Set \( n, m > 2 \) (if \( n = m = 1 \), the result is obvious).

From the Taylor expansion
\[
D^{n+1} u(x_0 + e) = D^{n+1} u(x_0) + \frac{1}{n!} D^n u(x_0) \cdot e + \frac{1}{(n+1)!} D^{n+1} u(x_0) \cdot e^2 + \ldots
\]
\[
= D^{n+1} u(x_0) + O\left(\frac{1}{e^{n+1}}\right)
\]
\[
D^{m+1} u(x_0 + e) = D^{m+1} u(x_0) + \frac{1}{(m+1)!} D^{m} u(x_0) \cdot e + \ldots
\]
\[
= D^{m+1} u(x_0) + O\left(\frac{1}{e^{m+1}}\right)
\]

If \( n = m \), we have
\[
Du(x_0 + e) = Du(x_0) + O\left(\frac{1}{e}\right)
\]
Then \( Du(x_0) \) has the singularity of \( O(1/e) \) and so \( u(x_0) \) has the singularity of \( O(\ln e) \).

If \( n < m \), we have
\[
u(x_0 + e) = u(x_0) + O\left(\frac{1}{e^{n-m}}\right)
\]
so \( u(x_0) \) has the singularity of \( O(1/e^{n-m}) \)

b) The result is obvious.

2. As mentioned above, the differential factor \( D^n \) for a given mechanical problem is determined by several constraints, and it may happen that \( a_i = n \) (\( i = 1, 2, \ldots, m \)). However, from the view of singularity analysis, the results in the theorem still holds. Below we give the conclusion in elasticity for the problem of singularity under a concentrated force.

(1) Elastic string

The equation is
\[
T \frac{d^2 u}{dx^2} = P(x)
\]