THE ELASTODYNAMIC SOLUTION FOR A SOLID SPHERE AND DYNAMIC STRESS-FOCUSBING PHENOMENON

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Abstract

This paper presents an analytical method of solving the elastodynamic problem of a solid sphere. The basic solution of the elastodynamic problem is decomposed into a quasi-static solution satisfying the inhomogeneous compound boundary conditions and a dynamic solution satisfying the homogeneous compound boundary conditions. By utilizing the variable transform, the dynamic equation may be transformed into Bassel equation. By defining a finite Hankel transform, we can easily obtain the dynamic solution for the inhomogeneous dynamic equation. Thereby, the exact elastodynamic solution for a solid sphere can be obtained. From results carried out, we have observed that there exists the dynamic stress-focusing phenomenon at the center of a solid sphere under shock load and it results in very high dynamic stress-peak.

Key words solid sphere, elastodynamics, dynamic stress-focusing

I. Introduction

The history and distribution of the dynamic stress in a solid sphere is a typical elastodynamic problem. The key to obtain the analytic solution for the problem lies in solving an elastodynamic basic equation with given boundary and initial conditions. Laplace transforms\(^1, 2\) can be used to solve the above problems but the solving process may be complex and sometimes, its inverse transforms are extremely difficult or even impossible. Thereby, to present a sufficient investigate for dynamic stress response in a solid sphere under shock load is still a very significant work. In this paper, we decompose the elastodynamic solution for a solid sphere into a quasi-static solution satisfying the inhomogeneous boundary conditions and a dynamic solution satisfying the homogeneous boundary conditions. By defining a finite Hankel transform, we can easily obtain the dynamic solution satisfying inhomogeneous equation with homogeneous boundary conditions. Finally, the exact solution for the problem is obtained.

In examples, we calculate the histories and distributions of dynamic stress in a solid sphere subjected sudden load. Through discussion, we see that the solution obtained presents the propagating features of the sphere wave. From the analytical expression of the solution and the results obtained, we can observe that there exists the phenomenon of dynamic stress-focusing at the center of a solid sphere under sudden load. The phenomenon result in
very high dynamic stress peak at the center. It is worth noting that the dynamic stress peak at the center is only finite value rather than singularity. The dynamic stress peak at the center and the dynamic stress response at other points in solid sphere oscillate dramatically as stress waves reflected successively from the free surface of the solid sphere.

The major accomplishment in this paper has been in gaining an effectively solving method and a better understanding of the dynamic stress-focusing effect at the center of a solid sphere under shock load.

II. Elastodynamic Equation and Solving Method

Consider that an elastic solid sphere of radius $b$ is subjected to a shock load. Its elastodynamic basic equation may be reduced to

$$
\frac{\partial^2 U(r,t)}{\partial r^2} + \frac{2}{r} \frac{\partial U(r,t)}{\partial r} - \frac{2}{r^2} U(r,t) = \frac{1}{V^2} \frac{\partial^2 U(r,t)}{\partial t^2} \quad 0 \leq r \leq b, \quad t > 0^+
$$

(2.1a)

where $U = U(r,t)$ expresses the radial displacement, $r$ and $t$ are the radial variable and the time variable respectively. $V = \sqrt{\frac{(\lambda + 2\mu)}{\rho}}$ is the wave speed, in which $\lambda$ and $\mu$ are Lame's constants and $\rho$ is the mass density.

Assume that a solid sphere possesses an arbitrary nonzero initial state at time $t = 0$. The initial conditions of the problem may be written as:

$$
U(r,0) = U_0(r), \quad U_r(r,0),, = V_0(r)
$$

(2.1b)

The radial displacement at $r = 0$ equals to zero and the radial stress at $r = b$ equals to a shock load $p(t)$, we have mixed boundary conditions

$$
\sigma_r(r,t) = \left[ (\lambda + 2\mu) \frac{\partial U}{\partial r} + \frac{2\lambda}{r} U \right]_{r=b} = p(t)
$$

(2.1d)

where $\sigma_r = \sigma_r(r,t)$ is the radial stress.

By applying displacement fields, geometry relations and constitutive relations, we can derive the expressions for dynamic stresses in a solid sphere as:

$$
\sigma_r(r,t) = (\lambda + 2\mu) \frac{\partial U}{\partial r} + \frac{2\lambda}{r} U
$$

(2.2a)

$$
\sigma_\varphi(r,t) = \sigma_r(r,t) = \lambda \frac{\partial U}{\partial r} + \frac{2(\lambda + \mu)}{r} U
$$

(2.2b)

The solution $U(r,t)$ for the basis equation (2.1) consists of a quasi-static solution $U_1(r,t)$ and a dynamic solution $U_2(r,t)$. Thus, we have

$$
U(r,t) = U_1(r,t) + U_2(r,t)
$$

(2.3)

where the quasi-static solution satisfies the following equation, and boundary conditions.

$$
\frac{\partial^2 U_1}{\partial r^2} + \frac{2}{r} \frac{\partial U_1}{\partial r} - \frac{2}{r^2} U_1 = 0
$$

(2.4a)

$$
U_1(0,t) = 0
$$

(2.4b)

$$
\left[ (\lambda + 2\mu) \frac{\partial U_1}{\partial r} + \frac{2\lambda}{r} U_1 \right]_{r=b} = p(t)
$$

(2.4c)