ON THE EXACT SOLUTION TO CERTAIN NON-LINEAR
PARTIAL DIFFERENTIAL EQUATIONS

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To be dedicated to my beloved teacher, H. Cartan, academician of French
Academy of Sciences, on the occasion of his 90th birthday.

Abstract

This paper, based on the theory of stratifications, gives a brand-new classification of partial differential equations.

Key words Janet number, bad system of equations, good system of equations

I. Introduction

It is widely known that the exact solution to certain partial differential equations plays an important role in the research of physics and mechanics problems. And for some equations, whose exact solution can not be obtained, the French mathematician E. Picard once introduced finite difference method, using discretized difference, to find its approximate solution. Since then, there have been important developments in the numerical calculation and its theoretical research. To name some, the theory of operators in functional space: the theory of pseudo Lie-group; Lagrang method: Cauchy Kowalewskaya theorem and a multitude of theorems of existence and uniqueness.

Unfortunately, all these studies failed to give a method to find the exact solution of a system of equations. Therefore, in seeking to find the solution of system of partial differential equations in engineering technique, physics or mechanics, the numerical calculation is still the most interesting method.

This paper is to discuss a new classification of partial differential equations and to answer the question of how to get the exact solution of certain equations.

To make things easy, the systems of equations mentioned in this paper all refer to those of non-linear partial differential equations.

II. Good System of Equations

As is the case in all mathematical research, what we are concerned with is the set of systems of partial differential equations in \( C^\infty \), denoting it as \( \mathcal{E} \).

Two aspects call for our attention. Firstly, R. Tham advocated\(^1\) that, for all the problems

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coming from nature, the class of analytic functions should be considered first. Secondly, according to the theory of J. Hadamard, if the unknown function of a system of equations is dependant on time, but any of its Cauchy problem on hyperplane \{t=t_0\} does not have well posed-initial-condition. Such a system of equations would not truly reflect the physical or mechanical phenomena.

Now, we consider two disjoint subsets of \( \mathcal{S} \), \( \mathcal{A} \) and \( \mathcal{R} \). They are defined as follows:

A system of equations \( D \in \mathcal{R} \) if and only if any solution of \( D \) and its restriction on any hypersurface will not define a well posed problem of \( D \). \( \mathcal{A} \) is the complementary set of \( \mathcal{R} \) in \( \mathcal{S} \).

So, according to J. Hadamard, the elements of \( \mathcal{R} \) are those systems of equations which are devoid of physical or mechanical import.

Therefore we call them bad systems of equations. This is why we introduce another subset of \( \mathcal{S} \) — "Good System of Equations".

A subset \( \mathcal{G} \subseteq \mathcal{S} \) is defined as follows:

A system of equations \( D \in \mathcal{G} \) if and only if \( D \) is a system of analytic equations and its Janet number \( i(D) \in \mathbb{R}^* \).

**Example 1** The following systems of equations are "Good System of Equations":
- Euler equation in fluid mechanics; Landau-Lifchitz equation (a complete system of equations for sticky, compressible fluid); system of elastic mechanic equations; Einstein equation in general theory of relativity; elliptical equation; wave equation; equation of heat conduction; Monge-Ampere equation.

**Theorem 1** If \( D \in \mathcal{G} \), then all the exact solutions of \( D \) can be expressed in the form of convergent series.

We would like to point out in particular that the systems of equations listed in Example 1 are all soluble in this meaning.

**Example 2** Consider the Euler equation of non-sticky, uncompressible fluid, the external force is \( F(x,t) = (x_1 x_3 t^2, t^2, t^3) \)

\[
\begin{align*}
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_1 \frac{\partial u_1}{\partial x_2} + u_1 \frac{\partial u_1}{\partial x_3} + \frac{1}{\rho} \frac{\partial p}{\partial x_1} - x_1 x_3 t^2 &= 0 \\
\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x_1} + u_1 \frac{\partial u_2}{\partial x_2} + u_1 \frac{\partial u_2}{\partial x_3} + \frac{1}{\rho} \frac{\partial p}{\partial x_2} - t^2 &= 0 \\
\frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x_1} + u_1 \frac{\partial u_3}{\partial x_2} + u_1 \frac{\partial u_3}{\partial x_3} + \frac{1}{\rho} \frac{\partial p}{\partial x_3} - t^3 &= 0 \\
\frac{\partial u_4}{\partial x_1} + \frac{\partial u_4}{\partial x_2} + \frac{\partial u_4}{\partial x_3} &= 0
\end{align*}
\]

We have its global solution in \( \mathbb{R}^4 \):

\[
\begin{align*}
u_1 &= c \quad (c \neq 0) \\
u_2 &= x_3 \left( \frac{1}{3c^2} x_1^3 - \frac{1}{c} x_3 t + t \right) \\
u_3 &= x_3 \left( \frac{1}{3c^2} x_1^3 - \frac{1}{c} x_3 t + t^2 + \frac{1}{51c} x_1^4 + \frac{2}{41c} x_3^2 t - \frac{1}{31} t^3 x_1^4 \right)
\end{align*}
\]