SINGULAR PERTURBATION OF GENERAL BOUNDARY VALUE PROBLEM FOR NONLINEAR DIFFERENTIAL EQUATION SYSTEM*

Zhou Yali (周雅丽) You Zhefeng (游哲丰) Lin Zongchi (林宗池)

(Received Sep. 6, 1996)

Abstract

In this paper, the singular perturbation of nonlinear differential equation system with nonlinear boundary conditions is discussed. Under suitable assumptions, with the asymptotic method of Lyusternik-Vishik and fixed point theory, the existence of the solution of the perturbation problem is proved and its uniformly valid asymptotic expansion of higher order is derived.

Key words nonlinear system, nonlinear boundary condition, singular perturbation, asymptotic expansion

I. Introduction

All physical systems are nonlinear to some extent. Actually, Linear system is imaginary model where nonlinear factor is omitted in nonlinear system. In solving the auto control, nonlinear oscillation theory, the boundary stagnation problem of fluid mechanics and some problems of semi-conduct theory and quantum mechanics etc., we only need to solve the following problem, which is nonlinear differential equation system with the small parameter in highest order derivative and nonlinear boundary conditions:

\[
\begin{align*}
    x' &= f(t,x,y,\varepsilon), \quad x(0,\varepsilon) = A(\varepsilon) \\
    y'' &= g(t,x,y,y',\varepsilon) \\
    h_1(y(0,\varepsilon),y'(0,\varepsilon),\varepsilon) &= B(\varepsilon) \\
    h_2(y(1,\varepsilon),y'(1,\varepsilon),\varepsilon) &= C(\varepsilon)
\end{align*}
\]  

where \( \varepsilon > 0 \) is a small parameter, \( t \in \mathbb{R}, x, f \in \mathbb{R}^n, y, g \in \mathbb{R}^m \), here \( \mathbb{R}^m, \mathbb{R}^n \) are \( m, n \) dimension real vector space with suitable normal, respectively.

Scholars at home and abroad have done some work on this sort of singular perturbation problem. For example, semilinear differential equation system with Dirichlet boundary value problem is discussed in paper [2]; Quasi-linear differential equation system with Dirichlet boundary value problem is discussed in papers [3~4]; Nonlinear differential equation system with Robin boundary value problem is discussed in paper [5]. Owing to their work, we can

---

* Project supported by the National Natural Science Foundation of Fujian Province
1 Quanzhou Liming Occupation University, Quanzhou 362000, P. R. China
2 Mathematics Department, Fujian Normal University, Fuzhou 350007, P. R. China

531
discuss the more general problem in this paper. We assume, that the following four conditions hold:

1. Reduced problem

\[ \begin{align*}
    x' &= f(t, x, y, 0), \quad x(0, 0) = A(0) \\
    y' &= g(t, x, y, r, 0), \quad h_1(y(0, 0), y(0, 0), 0) = B(0)
\end{align*} \]

there exists a solution \((X_0, Y_0) = (X_0(t), Y_0(t)) \in C^{(N+1)}[0, 1] \times C^{(N+2)}[0, 1]\) such that \(g_{xy}(t, X_0, Y_0, 0) > 0\). Simultaneously, for all \(\theta + Y_0(1)\) satisfying \(0 < \theta \leq\|Y_0(1)\|\) inner product\(\|\|\),

\[ \begin{align*}
    \theta^T \int_0^1 g_{xy}(1, X_0(1), Y_0(1) + r, Y_0(1) + r') ds > 0
\end{align*} \] (1.3)

here \(T\) represents the transpose and \(\|z\| = (z^T z)^{1/2}\);

2. \(f, g, h_1, h_2, A, B, C\) for every variable of them in the region of \(\Omega = \{0 \leq t \leq 1, \|x - X_0(t)\| \leq d_1, \|y - Y_0(t)\| \leq d_2, \|y' - Y_0(t)\| \leq d_3, 0 \leq \varepsilon \leq \varepsilon_0\}\) have continuous derivative of \(N+2\)th order and may expand according to the parameter of \(\varepsilon\), for example,

\[ A(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i A_i, \quad A_i = \left. \frac{1}{i!} \frac{d^i A(\varepsilon)}{d\varepsilon^i} \right|_{\varepsilon = 0} \]

3. \(h_2(Y_0(1), Y_0(1) - v_0(0), 0) = C_0, \quad v_0(+ \infty) = 0\) there exists a solution:

4. \(g_{xy}h_1 - g_{yx}h_1 \neq 0\).

II. Formal Asymptotic Solution

We assume that the system (1.1), (1.2) has a formal asymptotic solution:

\[ \begin{align*}
    x &= X(t, \varepsilon) + \varepsilon u(t, \varepsilon), \quad r = (1 - t)/\varepsilon \\
    y &= Y(t, \varepsilon) + \varepsilon v(t, \varepsilon)
\end{align*} \] (2.1)

where \(X(t, \varepsilon), Y(t, \varepsilon)\) are outer solution, \(u(t, \varepsilon), v(t, \varepsilon)\) are inner solution and have

\[ \begin{align*}
    (X(t, \varepsilon), Y(t, \varepsilon)) &\sim \left( \sum_{i=0}^{\infty} \varepsilon^i X_i(t), \sum_{i=0}^{\infty} \varepsilon^i Y_i(t) \right) \\
    (u(t, \varepsilon), v(t, \varepsilon)) &\sim \left( \sum_{i=0}^{\infty} \varepsilon^i u_i(t), \sum_{i=0}^{\infty} \varepsilon^i v_i(t) \right)
\end{align*} \]

We substitute (2.1), (2.2) into (1.1), (1.2) and have the following:

\[ \begin{align*}
    x' &= f(t, X, Y, \varepsilon), \quad X(0, \varepsilon) = A(\varepsilon) \\
    \varepsilon y' &= g(t, X, Y, Y', \varepsilon), \quad h_1(Y(0, \varepsilon), Y'(0, \varepsilon), \varepsilon) = B(\varepsilon) \\
    -u &= f(1 - \varepsilon r, X + \varepsilon u, Y + \varepsilon v, \varepsilon) - f(1 - \varepsilon r, X, Y, \varepsilon) \\
    v &= g(1 - \varepsilon r, X + \varepsilon u, Y + \varepsilon v, Y' - v, \varepsilon) - g(1 - \varepsilon r, X, Y, Y', \varepsilon) \\
    u(+ \infty) &= 0, \quad h_2(Y(1, \varepsilon) + \varepsilon v(0, \varepsilon), Y'(1, \varepsilon) - v(0, \varepsilon)) = C(\varepsilon)
\end{align*} \] (2.3)

where

\[ \begin{align*}
    \frac{du}{d\tau}, \frac{dv}{d\tau}
\end{align*} \]

We represent (2.3), (2.4) into power series of \(\varepsilon\) and let the coefficients of the some power of \(\varepsilon\)