THE HÖLDER CONTINUITY OF GENERALIZED SOLUTIONS OF
A CLASS QUASILINEAR PARABOLIC EQUATIONS

Wang Xiangdong (王向东)¹ Liang Xiting (梁廷廷)²

(Received May 3, 1996; Communicated by Zhang Shisheng)

Abstract

Let A and B satisfy the structural conditions (2), the local Hölder continuity
interior to \( Q = G \times (0, T) \) is proved for the generalized solutions of quasilinear
parabolic equations as follows:

\[
\frac{\partial u}{\partial t} - \text{div} A(x, t, u, \nabla u) + B(x, t, u, \nabla u) = 0
\]

Key words parabolic equation, natural growth condition, generalized
solution, local Hölder continuity

I. Introduction

Let \( E^n \) be the \( n \)-dimensional Euclidean space and \( G \) a bounded domain in \( E^n \), \( W^1_2(G) \) and
\( \dot{W}^1_2(G) \) be usual Sobolev spaces. Let \( 0 < T < \infty, p \geq 1 \). Denoted by \( L^p(0, T, L^2(G)) = \{ u; u:(0, T) \rightarrow L^2(G) \} \) be Banach space with norms
\[
\| u \|_{L^p(0, T, L^2(G))} = \left( \int_0^T \| u \|_{L^2(G)}^p \, dt \right)^{1/p}
\]

\[\| u \|_{W^1_p(0, T, L^2(G))} = \text{max} \| u \|_{L^2(G)} \]

\( W^1_2(0, T, L^2(G)) \) etc. are all the same. Denoted by \( Q = G \times (0, T) \). Consider the following
quasilinear parabolic equation:

\[
\frac{\partial u}{\partial t} - \text{div} A(x, t, u, \nabla u) + B(x, t, u, \nabla u) = 0 \quad (1.1)
\]

where \( u_t = \partial u / \partial t, \nabla u = (\partial u / \partial x^a, a = 1, 2, \cdots, n), A(x, t, u, \xi) \) and
\( B(x, t, u, \xi) \) are Caratheodory functions defined on \( Q \times E^1 \times E^n \), i.e. \( A \) and \( B \) are continuous in \( u \) and \( \xi \) for
fixed \( x \) and \( t \) in \( G \) (almost everywhere) and measurable in \( x \) and \( t \) for fixed \( u \) and \( \xi \).

Moreover, suppose \( A \) and \( B \) satisfy the following structural conditions:

\[
\begin{align*}
\nabla u \cdot A(x, t, u, \nabla u) & \geq \| \nabla u \|^2 - \kappa_1 \| u \|_l^2 - f_0(x, t) \\
| A(x, t, u, \nabla u) | & \leq \kappa_1 | \nabla u | + \kappa \| u \|_l^{1/2} + f_1(x, t) \\
| B(x, t, u, \nabla u) | & \leq C(x, t) | \nabla u |^\gamma + \kappa \| u \|_l^{1-l+1} + f_2(x, t)
\end{align*}
\]

(1.2)

where \( \kappa \geq 0, \kappa_1 \geq 1, l = 2(1 + 2/n) \) and \( 1 \leq \gamma \leq 2 \) are constants and \( C(x, t) \) and

¹ Department of Basic Courses, Zhengzhou Institute of Light Industry, Zhengzhou 450002, P. R. China
² Department of Mathematics, Zhongshan University, Zhongshan 510275, P. R. China
$f_i(x,t) \ (i = 1,2)$ satisfy respectively

$$C(x,t) \in L^r(Q)$$

$$f_i(x,t) \in L^q_i(Q) \quad (i = 1,2)$$

$$S_0, S_2 > (n + 2)/2, \ S_1 > n + 2$$

As usual $u \in L^w(0,T,L_2(G)) \cap L^q(0,T,W^1_2(G))$ is called a generalized solutions of (1.1) if it satisfies

$$\int_0^T \int_G \{-u_t + \nabla \cdot A(x,t,u) - \nabla \cdot B(x,t,u) + v B(x,t,u,\nabla u) + v B(x,t,u,\nabla u)\} \ dx \ dt = 0$$

$$(\forall t \in (0,T), v \in W^1_2(0,T,L_2(G)) \cap L^q(0,T,W^1_2(G)))$$

If suppose $u \in L^r(\mathcal{Q})$ and $t^*$ satisfying the following conditions

$$r > (n + 2)/(2 - \gamma)$$

and

$$t^* = \begin{cases} \frac{r(n + 2)(\gamma - 1)}{2r - 2 - \gamma}, & \text{as } \frac{n + 2}{2 - \gamma} < r < +\infty \\ \frac{(n + 2)(\gamma - 1)}{2 - \gamma}, & \text{as } r = +\infty \end{cases}$$

then the local boundedness has been proved of generalized solution of (1.1) in [2]. In the following we shall prove the local Hölder continuity of general solutions of (1.1). We have following

**Theorem** Let $u \in L^w(0,T,L_2(G)) \cap L^q(0,T,W^1_2(G)) \cap L^w(Q)$ be a generalized solution of equation (1.1). Suppose conditions (1.2)−(1.5) are fulfilled and $\gamma \in (1 + 2/ (n + 2),2)$ is in condition (1.2). Then $u$ is local Hölder continuity in $G$.

Denoting $B(x_0, \rho) = \{x \in E^d; \ |x - x_0| < \rho \}$ and $B(\rho) = B(0, \rho)$. In order to prove the theorem we need the following Lemma:

**Lemma** Let $S$ be a subset of $B(\rho)$ with positive measure. Let $u \in W^1_q(B(\rho))$ and $u = 0$ on $S$. Let $\eta(x) = \eta(\ |x\ |)$ be a piecewise linear continuous function of $\ |x\ |$ and suppose $0 \leq \eta(x) \leq 1$ and $\eta(x) = 1$ on $S$. Then for any $e \subset B(\rho)$ (e be measurable), such that

$$\int_{\eta_e} (\ |u(x)| \ |\nabla u(x)\ | \eta(x) \ dx) \leq C(n)\rho^{n-1} \int_{B(\rho)} (\ |u(x)| \ |\nabla u(x)\ | \eta(x) \ dx)$$

**II. Proof of Theorem**

We confine ourselves $\kappa = 0$ and $u \in L_2(Q)$ as follows on account of supposing $u$ to be bounded in $G$ and [1]. We might assume $B(\rho) \times (0,\rho^2) \subset Q$. Let $\zeta(x) = \zeta(\ |x\ |)$ be a piecewise linear continuous function of $\ |x\ |$ defined by

$$\zeta(x) = \begin{cases} 1, & \text{as } \ |x\ | \leq (1 - \lambda)\rho \\ (\lambda\rho)^{-1}(\rho - \ |x\ |), & \text{as } (1 - \lambda)\rho < \ |x\ | < \rho \\ 0, & \text{as } \ |x\ | \geq \rho \end{cases}$$

where $\lambda \in (0,1)$ is constant. Therefore for any $k \in (-M,M)$ we may take $v = \zeta^2(x)$