INFINITELY MANY SOLUTIONS FOR DOUBLE HARMONIC PERTURBED PROBLEM

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(Received Nov. 22, 1993; Communicated by Zhang Hongqing)

Abstract
In this paper we consider the double harmonic perturbed problem on a bounded domain with boundary-value zero. The results which we have obtained have improved the results obtained in [1], [3] and [4].

Key words double harmonic operator, generalized Morse indice, nontrivial solution

I. Introduction
Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), we consider the double harmonic perturbed problem,

\[
\begin{align*}
\Delta^2 u - \alpha \Delta u + bu &= g(x, u) + f(x, u) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \alpha \geq 0 \), \( b \geq 0 \), \( n \) is the out normal direction of \( \partial \Omega \), \( f(x, u) \) and \( g(x, u) \) satisfy the following conditions:

\begin{enumerate}
\item \( g(x, s) \in C(\overline{\Omega} \times \mathbb{R}) \), and for any \( x \in \Omega \), \( g(x, s) \) is \( C^1 \) and odd with respect to \( s \).
\item There are constants \( \mu \in (0, 1) \), \( s_0 > 0 \) such that:
\[
0 < \frac{g(x, s)}{s} < \mu g^1(x, s) \quad \text{for } s \geq s_0, \, x \in \overline{\Omega}.
\]
\item \( \lim_{s \to +\infty} \frac{g(x, s)}{s^\rho} = q(x) > 0 \) uniformly with respect to \( x \) and \( q(x) \) is continuous.
\item \( f(x, s) \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}) \), \( f(x, s) \) is \( C^1 \) with respect to \( s \).
\item \( \lim_{s \to +\infty} \frac{f^1(x, s)}{g^1(x, s)} = l \) uniformly with respect to \( x \), where \( l < 1 \), \( f^1 = \frac{\partial f}{\partial s}, \; g^1 = \frac{\partial g}{\partial s} \).
\item \( F(x, s) = \int_0^s f(x, t) \, dt \) is uniformly bound on \( x \) and \( s \).
\end{enumerate}

Problem (I) has been studied by many authors. See [1], [3], [4], etc. In this paper, by means of the perturbation method developed in [1] and Harmonic analysis and Morse theory we have obtained a relatively perfect multiple solutions result as follows.

Theorem 1 Suppose \( a \geq 0, \; b \geq 0 \), \( g(x, u) \) satisfies \((g_1) - (g_4)\), \( f(x, u) \) satisfies \((f_1) - (f_2)\) and \( p \) satisfies:
\[ 1 < p < \frac{N + 4}{N - 4}, \quad \text{for} \ N \geq 5, \]
\[ 1 < p < \frac{N + 2}{N - 2}, \quad \text{for} \ N = 3 \text{ or } 4, \]
\[ 1 < p < +\infty, \quad \text{for} \ N \leq 2. \]

Then problem (I) has infinitely many nontrivial solutions. This result has improved the results obtained in [1], [3], [4].

II. Preliminary

Let \( v(x) \geq 0 \), In this section we consider the following eigenvalue problem:

\[
\begin{align*}
\Delta^2 u - v(x) u &= \lambda u & \text{in} & \Omega \\
\frac{\partial u}{\partial n} &= 0 & \text{on} & \partial \Omega
\end{align*}
\]

If \( M(v) \) denotes the number of nonpositive eigenvalues for problem (II) in \( H^1_0(\Omega) \), then we can prove that

**Theorem 2** There exists a positive constant \( c \) depending on \( \Omega \) and \( N \) only, such that

\[
M(v) \leq c \| v(x) \|_{L^N}^N \quad \text{for} \ N \geq 5 \quad \text{and} \quad v(x) \in L^N(\Omega)
\]
\[
M(v) \leq c \| v(x) \|_{L^N}^N \quad \text{for} \ N = 3, 4 \quad \text{or} \quad v(x) \in L^N(\Omega)
\]
\[
M(v) \leq c \| v(x) \|_{L^N}^N \quad \text{for} \ N \leq 2 \quad \text{and} \quad v(x) \in L^N(\Omega)
\]

To prove theorem 2, we give some Lemmas as follows.

**Lemma 1** ([6]) \( M(v) \) is equal to the number of eigenvalues less than 1 for the following eigenvalue problem

\[
\begin{align*}
\Delta^2 u - v(x) u &= \lambda u & \text{in} & \Omega \\
\frac{\partial u}{\partial n} &= 0 & \text{on} & \partial \Omega
\end{align*}
\]

**Lemma 2** ([6]) If \( N \geq 5 \), \( v(x) \in L^N(\Omega) \), then there exists a positive constant \( c \) such that:

\[
M(v) \leq c \| v(x) \|_{L^N}^N.
\]

If \( M(c, v) \) denotes the number of eigenvalues less than \( c \) for eigenvalue problem

\[
\begin{align*}
-\Delta u - v(x) u &= \lambda v(x) u & \text{in} & \Omega \\
\frac{\partial u}{\partial n} &= 0 & \text{on} & \partial \Omega
\end{align*}
\]

then we have

**Lemma 3** There is a positive constant \( c_0 \) such that \( M(v) \leq M(c_0, v) \).

**Proof** There is a positive constant \( c \) such that