NUMERICAL SOLUTION OF THE SINGULARLY PERTURBED PROBLEM WITH NONLOCAL BOUNDARY CONDITION*

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Abstract: Singularly perturbed boundary value problem with nonlocal conditions is examined. The appropriate solution exhibits boundary layer behavior for small positive values of the perturbative parameter. An exponentially fitted finite difference scheme on a non-equidistant mesh is constructed for solving this problem. The uniform convergence analysis in small parameter is given. Numerical example is provided, too.

Key words: exponentially fitted difference scheme; singular perturbation; nonlocal boundary condition

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Introduction

This paper is concerned with the numerical solution, by finite difference method, of the following singularly perturbed nonlocal boundary value problem:

\[ Lu = -\varepsilon u'' + a(x)u = f(x) \quad (0 < x < l), \]  
\[ L_0 u = -\sqrt{\varepsilon} u'(0) + \gamma u(0) = \mu_0, \]  
\[ L_1 u = u(l) - \delta u(d) = \mu_l \quad (0 < d < l), \]  

where \( \varepsilon \) is a small positive parameter, \( \gamma > 0, \delta, \mu_0 \) and \( \mu_l \) are given constants, \( a(x) \geq a > 0 \) and \( f(x) \) denote sufficiently smooth real functions of \( x \), so that a unique solution \( u(x) \) exists for all small \( \varepsilon \) values. This solution has in general a boundary layers at \( x = 0 \) and \( x = l \) for \( \varepsilon \) near 0 (see Section 1).

Singularly perturbed differential equations (differential equations with a small parameter \( \varepsilon \) multiplying the highest order derivative) are certainly of interest in many scientific and engineering applications. Among these are the Navier-Stokes equations of fluid flow at high Reynolds number\(^{1,2} \), mathematical models of liquid crystal materials and chemical reactions\(^{3} \), control theory\(^{4} \), electrical networks\(^{2,5} \).

The use of classical numerical methods for solving such problems may give rise to difficulties...
when the perturbation parameter $\varepsilon$ is small \cite{1,6,2,7,8}. Therefore, it is important to develop suitable numerical methods to these problems. There is a vast literature dealing with numerical methods for Eq. (1) with two-point type boundary condition, e.g., \cite{1,6,2}. Several schemes on an equidistant mesh for numerical solution of Eq. (1) with one nonlocal boundary condition and with first type boundary condition have been proposed in \cite{9,10} (For references on nonlocal boundary value problems and their applications see example \cite{11,12}). Recently some new perturbation techniques are proposed by He \cite{13-19}, which are valid not only for small parameter, but also for large values of the parameter. A review of recently developed analytical techniques is also given by He in Ref. \cite{17}.

In this note we present uniformly convergent difference scheme on a non-equidistant mesh for the numerical solution of the problems (1) – (3). Here we are interested in robust methods that work for all values of the perturbation parameter. Our approach construct difference schemes is based on the method of integral identities by using an exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form \cite{20-22}. In the discrete maximum norm uniform error estimates are established. A numerical example is also considered.

We shall use the notation of \cite{23} for mesh functions.

1 Preliminary Results

Here we establish the asymptotic estimations of the problems (1) – (3) that are needed in later sections for the analysis appropriate numerical solution.

**Lemma 1** Let $u(x)$ be the solution of the problems (1) – (3) and assume that $a, f \in C^1[0, l]$.

Moreover

$$1 - \delta u_1(d) \neq 0, \quad (4)$$

where $u_1(x)$ is the solution of the two-point boundary value problem

$$L u_1 = 0 \quad (0 < x < l), \quad (5)$$

$$L_0 u_1 = 0, \quad u_1(l) = 1. \quad (6)$$

Then the estimates

$$\| u \|_{C[0, l]} \leq C, \quad (7)$$

$$1 \leq \frac{\| u' \|}{\| u \|} \leq C \left\{ 1 + \frac{1}{\sqrt{\varepsilon}} \exp \left[ \frac{\sqrt{a(x)}}{\sqrt{\varepsilon}} \right] + \exp \left[ \frac{\sqrt{a(l - x)}}{\sqrt{\varepsilon}} \right] \right\} \quad (0 \leq x \leq l) \quad (8)$$

hold, where $C$ is a positive constant independent of $\varepsilon$ (also of mesh size, in our discussion about numerical solution).

**Proof** Let $u(l) = \lambda$. Then $u(x)$ may be represented in the form

$$u(x) = u_0(x) + \lambda u_1(x), \quad (9)$$

where $u_0(x)$ is the solution of the two-point boundary value problem

$$L u_0 = f(x) \quad (0 < x < l), \quad (10)$$

$$L_0 u_0 = \mu_0, \quad u_0(l) = 0. \quad (11)$$

Eqs. (3) and (9) then imply