MINIMAL AND MAXIMAL FIXED POINT THEOREMS AND ITERATIVE
TECHNIQUE FOR NONLINEAR OPERATORS IN PRODUCT SPACES*

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Abstract

In this paper, we study minimal and maximal fixed point theorems and iterative
 technique for nonlinear operators in product spaces. As a corollary of our result, some
coupled fixed point theorems are obtained, which generalize the coupled fixed point
theorems obtained by Guo Da-jun and Lankshmikantham and the results obtained by Lan
in [4], and [6].

Key words minimal and maximal fixed point, coupled fixed point

The monotone iterative technique is an important method for studying the solutions of
nonlinear differential equations (see [1], [2]). On the basis of coupled quasisolutions of the initial
value problems for ordinary differential equations, Guo and Lakshmikantham in [2] introduced
the concept of abstract coupled fixed points for some operators and obtained many coupled fixed point
theorems and their applications. In this paper, we first obtain a minimal and maximal fixed point
theorem for nonlinear operators in product spaces, which extends Theorem 1 in [4] and Theorem 1
in [5]. As a corollary of our result, a coupled fixed point theorem for nonlinear operators is
obtained, which extends Theorem 1 in [2] and Theorem 1 in [6].

I. Statement and the Proofs of Theorems

Let X be a real Banach space, P a cone in X, and "≤" an ordering in X induced by P defining y≥x
if and only if y-x∈P. The pair (X,P) is called an ordered Banach space with positive cone P, if X is
given ordering induced by P. Let u_0, v_0∈X, and u_0<v_0 [u_0<v_0]={x∈X, u_0≤x≤v_0}. An ordered
sequence {x_n} in X is said to be nondecreasing (resp. nonincreasing) if, for each n∈N (natural
number set), x_n≤x_{n+1} (resp. x_n≥x_{n+1}). A map A:D⊂X→X is said to be nondecreasing (resp.
nonincreasing) if x≤y (x,y∈D) implies Ax≤Ay (resp. Ay≥Ax). If D⊂X is a bounded set, then the
set measure of noncompactness of D, ν(D) is defined by

ν(Q)=inf{d>0: D=∪_{i=1}^{m} D_i, diam(D_i)≤d} (1.1)

Clearly, ν(Q)=0 if and only if Q is precompact, and ν(D_1∪D_2)=max{ν(D_1), ν(D_2)} other
properties can be seen in [3]. A map A:D⊂X→X is said to be condensing if, ν(A(Q))<ν(Q) for
any bounded set Q⊂D with ν(Q)=0. Obviously, the completely continuous map A is condensing.
Let → and ← denote strong and weak convergence, respectively.

Lemma 1 Let (X,P) be an ordered Banach space and {u_n} a monotone sequence (i.e. {u_n} is

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nondecreasing or nonincreasing). If \( \{u_{n_k}\} \subseteq \{u_n\} \), \( \{u_{m_i}\} \subseteq \{u_n\} \) and \( u_{n_k} \to u^* \), \( u_{m_i} \to u^* \), then \( u^* = u^* \).

**Proof** Using the monotoneness of \( \{u_n\} \) and weak closeness of \( P \). Lemma 1 can be easily proved.

**Lemma 2** Let \((X, P)\) be an ordered Banach space. If \( \{u_n\} \) is a weakly precompact monotone sequence in \( X \), then there exists \( u \in X \) such that \( u_n \to u \). Furthermore, assume \( \{u_n\} \) is nondecreasing (resp. nonincreasing) then \( u_n \leq u \) (resp. \( u_n \geq u \)) for \( n \in \mathbb{N} \).

**Proof** Without loss of generality, we may assume that \( \{u_n\} \) is a nondecreasing sequence. Since \( \{u_n\} \) is weakly precompact, there exist \( \{u_{n_k}\} \subseteq \{u_n\} \) and \( u_0 \in X \) such that \( u_{n_k} \to u_0 \). If \( u_n \to u_0 \), then there exist \( \varepsilon \geq 0 \) and \( f_0 \in X^* \) (dual space of \( X \)) and \( \{u_{n_k}\} \subseteq \{u_n\} \) such that \( |f_0(u_{n_k} - u_0)| \geq \varepsilon \). Also since \( \{u_{n_k}\} \) is weakly precompact, there exists a weakly converging subsequence \( \{u_{m_i}\} \subseteq \{u_{n_k}\} \), we may assume \( u_{m_i} \to u^* \). So we see \( u^* = u_0 \) which contradicts Lemma 1. Hence \( u_n \to u_0 \). Since \( u_{n+m} \geq u_n \) and \( u_{n+m} = u_n \in P \) for any fixed \( n \in \mathbb{N} \) and any \( m \in \mathbb{N} \) and \( P \) is weakly closed, \( m \to +\infty \) yields \( u_0 = u \in P \), and thus, \( u_n \leq u_0 \) for any \( n \in \mathbb{N} \).

Let \( X \) and \( Y \) be two real Banach spaces with norm \( \| \cdot \|_X \) and \( \| \cdot \|_Y \), respectively. The norm in product space \( X \times Y \) is defined as follows: For \( 1 \leq p \leq +\infty \) and \((x, y) \in X \times Y \), set

\[
\| (x, y) \| = \begin{cases} \max \{ \| x \|_X, \| y \|_Y \} & (p = +\infty) \\ \left( \| x \|_X^{\frac{1}{p}} + \| y \|_Y^{\frac{1}{p}} \right)^{\frac{1}{1/p}} & (1 \leq p < +\infty) \end{cases}
\]

(\( * \))


(\( ** \))

It is known that \( X \times Y \) is a Banach space with the norm defined above and with linear space addition and scalar multiplication defined componentwise, and

\[
(x_n, y_n) \to (x_0, y_0) \text{ if and only if } x_n \to x_0 \text{ and } y_n \to y_0
\]

(1.2)

**Lemma 3** Let \((X, P_1)\) and \((Y, P_2)\) be two ordered Banach spaces. Then (1.1) \((X \times Y, P_1 \times (-P_2))\) is an ordered Banach space with positive cone \( P_1 \times (-P_2) \). The order in the following product space is induced by \( P_1 \times (-P_2) \).

(2) Let \( w_1 = (x_1, y_1) \), \( w_2 = (x_2, y_2) \) and \( w_1, w_2 \in X \times Y \), then \( w_1 \preceq w_2 \) if and only if \( x_1 \preceq x_2 \) and \( y_2 \preceq y_1 \).

Now we prove the main result of this paper.

**Theorem** Let \((X, P_1)\) and \((Y, P_2)\) be two ordered Banach spaces and \((u_0, v_0), (x_0, y_0) \in X \times Y \) such that \( (u_0, v_0) \preceq (x_0, y_0) \). Suppose that \( B : (u_0, v_0), (x_0, y_0) \mapsto X \times Y \) is nondecreasingly condensing map. If the following conditions hold:

\((H_1)\) \( B(D) \) is bounded;

\((H_2)\) \( (u_0, v_0) \preceq B(u_0, v_0) \) and \( B(x_0, y_0) \preceq (x_0, y_0) \);

\((H_3)\) If \( x_n \to x \), then for any \( y \in Y \), \( B(x_n, y) \to B(x, y) \) and if \( y_n \to y \), then for any \( x \in X \), \( B(x, y_n) \to B(x, y) \).

Then (a) \( B \) has minimal and maximal fixed points \( (u^*, v^*) \) and \( (x^*, y^*) \) i.e. \( B(u^*, v^*) = (u^*, v^*) \), \( B(x^*, y^*) = (x^*, y^*) \) and for any fixed point \((x, y) \in (u_0, v_0), (x_0, y_0) \) of \( B \), \( u^* \preceq x \preceq x^* \) and \( y^* \preceq y \preceq y^* \). Moreover, we have

\[
(\text{b) } u^* = \lim_{n \to +\infty} u_n, \quad v^* = \lim_{n \to +\infty} v_n, \quad x^* = \lim_{n \to +\infty} x_n \quad \text{and} \quad y^* = \lim_{n \to +\infty} y_n, \quad \text{where } \quad (u_{n+1}, v_{n+1}) = B(u_n, v_n) \quad \text{and} \quad (x_{n+1}, y_{n+1}) = B(x_n, y_n), \quad n = 0, 1, 2, \ldots \text{ and}\n\]

\[
(\text{c) } u_0 \preceq u_1 \preceq \ldots \leq u_n \preceq \ldots \leq x_n \preceq \ldots \leq x_0 \quad \text{and} \quad y_0 \preceq y_1 \preceq \ldots \preceq y_n \preceq \ldots \preceq y_0.
\]