AN EXTREMUM THEORY OF THE RESIDUAL FUNCTIONAL
IN SOBOLEV SPACES $W^{m,p} (\Omega)$

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Abstract

In the present paper the concept and properties of the residual functional in Sobolev space are investigated. The weak compactness, force condition, lower semi-continuity and convex of the residual functional are proved. In Sobolev space, the minimum principle of the residual functional is proposed. The minimum existence theorem for $J(u) = 0$ is given by the modern critical point theory. And the equivalence theorem or five equivalence forms for the residual functional equation are also proved.

Key words Sobolev spaces, residual functional, infinite Banach spaces, convex, lower semi-continuity, force condition, minimum existence theorem

I. Introduction

Since the 3rd National Conference on the Method of Weighted Residual (MWR) was held, the MWR has had important applications in wide aspects of applied mathematics and calculating mechanics\(^1,^2\). Qiu proposed a strong and weak dual space principle in Hilbert space and proved that the uniform convergence of MWR is true\(^3\). Ling\(^4,^5\) proved the existence of solutions to one class of the nonlinear differential equation, proposed the residual inequality, and gave the estimate of the error bounds and the convergence of MWR in norm sense. But the theoretical foundations of mathematics of MWR have not yet been established systematically.

The studies of mathematical theory of the residual functional in Sobolev space $W^{m,p} (\Omega)$ are given in this paper. The properties of the residual functional, the minimum principles of the residual functional in Sobolev space and the minimum existence theorem are proposed, and now the method of weighted residual has been based on the Sobolev space theory.

II. The Concept of the Residual Functional

Let us consider that the properly posed problem [7] of the partial differential equation (2.1) and the boundary value (2.2) in Sobolev space as follows:

\[
\begin{align*}
\begin{bmatrix} L(u) = f(x) \\ B(u) = g(x) \end{bmatrix} & \quad (\forall x \in \Omega \subseteq \mathbb{R}^n) \\
(2.1) & \\
(2.2) & 
\end{align*}
\]

where the partial differential operator

\[
L(u) = \sum_{|\alpha| = m}^{} A_{\alpha} \frac{\partial^{1,\alpha} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} + F(x, u, Du, \cdots, D^{m-1}u)
\]

\[x = (x_1, \cdots, x_n); |\alpha| = \alpha_1 + \cdots + \alpha_n, \alpha_i \geq 0, \quad i = 1, \cdots, n \quad (2.3)\]
The coefficients $A_{i}(x)$ of the derivatives of order $m$ in operator (2.3) have properly differentiable and algebraic behavior. The nonlinear term $F$ is supposed to be a function whose derivatives are lower than order $m$. And the coefficients $A_{i}(x, u, Du, \cdots, D^{m-1}u)$ in (2.4) express the $L_{i}$ is a quasi-linear operator. The partial differential equation (2.1) in which $L(u)$ is (2.3) or (2.4) is usually called nonlinear partial differential equation.

In Sobolev space, we consider the solution of the properly posed problem [I], $u(x) \in W^{m,p}(\Omega)$. The $\Omega$ is a bounded open domain or set in the $n$-dimensional vector space $\mathbb{R}^{n}$ and its measure $\Omega > 0$. In order that the embedding theorem holds true, the domain $\Omega$ has the strong local Lipschitz conditions or a cone domain or $L$-domain. The boundary of $\Omega$, $\partial \Omega$ is composed of finite super-surface $S_{i}$(n-1-dimension) which is sufficiently smooth. $B(u)$ is a linear partial differential operator on $\partial \Omega$ and its highest order is of $m-1$. And condition (2.2) may contain an initial condition. The nonhomogeneous term $f(x) \in W^{m,p}(\Omega) = L^{r}(\Omega)$ and $g(x) \in W^{m,p}(\partial \Omega) = L^{r}(\partial \Omega)$.

1. Definitions and elementary properties of a residual functional

A residual function of equation (2.1) or (2.2) is defined by $u \in W^{m,p}(\Omega)$, such as

$$R(u) = (L(u) - f) \in L^{r}(\Omega) \quad (\forall u \in W^{m,p}(\Omega))$$

$$R_{B}(u) = (B(u) - g) \in L^{r}(\partial \Omega) \quad (\forall u \in W^{m-1,p}(\partial \Omega))$$

The function $R(u) \in L^{r}(\Omega)$ or $R_{B}(u) \in L^{r}(\partial \Omega)$ is sometimes called the residual function of the partial differential operator $L$ or $B$.

**Definition 1** A Residual Functional of [I] is

$$J(u) = ||R(u)||_{L^{r}(\Omega)} + ||R_{B}(u)||_{L^{r}(\partial \Omega)} \quad (1 \leq p; ||R(u)||_{L^{r}(\Omega)} = J(R(u)) < +\infty$$

**Definition 2** A residual functional of equation (2.1) is

$$J(u) = ||R(u)||_{L^{r}(\Omega)} < +\infty \quad (1 \leq p; \forall u \in W^{m,p}(\Omega))$$

From definition (2.8) or (2.7), its minimum is zero. To seek the minimum of (2.8) or (2.7) is then changed to solve the functional equation $J(u) = 0$. We have five elementary properties as follows:

1) $J(u) = ||R(u)|| \geq 0$

2) $\min J(u) = 0$, $\forall u \in U \subset W^{m,p}(\Omega)$, if the minimum of $J(u)$ exists.

3) mapping $J: W^{m,p}(\Omega) \to \mathbb{R}$