INTEGRABLE TYPES OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS OF HIGHER-ORDERS

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Abstract

In this paper, some integrable types of more general nonlinear ordinary differential equations of higher-orders are proposed in virtue of Leibnitz formula, and formulas of higher-order derivatives of the composite functions as well as substitution variables. The expressions for the general integrations of some of the equations are presented. The results obtained are the generalization of those in the references. Finally, some examples are also given.

Key words: nonlinear differential equation of higher order, integrable kind, general integral

I. Theorems and Corollaries

Theorem 1 Suppose \( y = y(x) \), \( w(y), f(x) \in C^n, Q(x) \in C \), and \( w'(y) \neq 0 \), \( f(x) \neq 0 \) then the nonlinear equation of order \( n \) below is integrable.

\[
\sum_{k=1}^{n-1} c_k \frac{f^{(k)}(y)}{(n-k)!} \left( \frac{y^{(n-k)}}{(n-k-1)!} \right) \ldots = Q
\]

where \( f^{(k)}(x), w^{(k)}(y), y^{(k)}(x) \), are the \( k \)-th derivative of \( f(x), w(y), \) and \( y(x) \) with respect to their own variable

and the general integration of (1.1) is

\[
w(y) = \frac{1}{f} \left[ c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n + \int \left( \int \cdots \int Q \, dx \right) dx \ldots \right]
\]

where \( c_i (i = 1, 2, \ldots, n) \) are the arbitrary constants.

Proof Let \( u = w(y) \). Using the higher-order derivative formula of the composite function (see [1], pp. 197):
\[
\frac{d^n}{dx^n}[w(y(x))] = \sum_{1 \leq i_1 < \cdots < i_n \leq m} \frac{n!}{i_1! \cdots i_n!} \left( \frac{y^{(i_1)}}{1!} \right)^{i_1} \cdots \left( \frac{y^{(i_n)}}{1!} \right)^{i_n}
\]

we can reduce equation (1.1) to

\[\sum_{l=0}^{n} C_l f^{(l)} u^{(n-l)} = Q.\]

which can be further reduced to

\[-(fu)^{(n)} = Q\]  \hspace{1cm} (1.3)

by Leibnitz formula ([1], PP. 197)

\[(fu)^{(n)} = \sum_{l=0}^{n} C_l f^{(l)} u^{(n-l)}\] \hspace{1cm} (B)

and (1.3) is equivalent to (1.1).

The general solution of equation \((fu)^{(n)} = 0\) is

\[u_0 = \frac{1}{f} \left( c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n \right)\] \hspace{1cm} (1.4)

A particular solution of equation \((fu)^{(n)} = Q\) is

\[u = \frac{1}{f} \int dx \int dx \cdots \int Q dx\] \hspace{1cm} (1.5)

Hence \(u = u_0 + \bar{u}\) is the general solution of (1.3); \(w(y) = u\) is the general integration of (1.1). From (1.4) and (1.5), we have (1.2). The proof of the theorem is completed.

When \(n = 3\) in the theorem 1, we have

**Corollary 1** Suppose \(y = y(x), w(y), f(x) \in \mathcal{C}^3, Q(x) \in \mathcal{C}, \) and \(w'(y) \neq 0, f(x) \neq 0\), then the 3rd nonlinear equation

\[
\frac{f^3}{y^3} w^y(y) + [3f y^y + 3f' y^2] w^y(y) + [f y^y + 3f' y^2 + 3f'' y] w'(y) + f^y w(y) = Q
\]

is integrable, and its general integration is

\[w(y) = \frac{1}{f} \left[ c_1 x^2 + c_2 x + c_3 + \int dx \int dx \cdots \int Q dx \right].\]

**Theorem 2** Suppose \(y = y(x), w(y) \in \mathcal{C}^{n+1}, f(x), g(x), h(x) \in \mathcal{C}^n, Q(x) \in \mathcal{C}\) and \(w'(y) \neq 0, f(x) \neq 0\), then the nonlinear equation of order \((n+1)\)

\[
f \sum_{1 \leq j_1 < \cdots < j_{n+1} \leq m+1} \frac{(n+1)!}{j_1! \cdots j_{n+1}!} \left( \frac{y^{(j_1)}}{1!} \right)^{j_1} \cdots \left( \frac{y^{(j_{n+1})}}{1!} \right)^{j_{n+1}}
\]

\[
\sum_{1 \leq k_1 = j, \Sigma k_j = n+1}
\]

\[
\in \{1, 2, \ldots, m+1\}
\]