NUMERICAL METHODS FOR PARABOLIC EQUATION WITH A SMALL PARAMETER IN TIME VARIABLE

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Abstract

In this paper, we discuss the parabolic equation with a small parameter on the derivative in time variable. We construct difference scheme on the non-uniform mesh according to Bakhvalov, and prove the one-order uniform convergence of this scheme. Numerical results are presented.

Key words difference scheme, uniform convergence, parabolic type equation, non-uniform mesh

I. Introduction

In this paper we discuss a parabolic equation with a small parameter in time variable,

\[ Lu = a(x,t) \frac{\partial^2 u}{\partial x^2} + b(x,t) \frac{\partial u}{\partial x} - c(x,t) u - \varepsilon \frac{\partial u}{\partial t} = f(x,t,\varepsilon), \quad (x,t) \in D \]

\[ u(x,0) = \varphi(x), \quad u(0,t) = \psi_0(t), \quad u(1,t) = \psi_1(t) \]

where \( D = \{0 < x < 1, \ 0 < t < T\} \), and \( a, b, c, f, \varphi, \psi_0 \) and \( \psi_1 \) are sufficiently smooth with

\[ a(x,t) \geq \alpha > 0, \ c(x,t) \geq \varepsilon > 0 \]  

on \( D \). Titov[1] previously constructed the exponentially fitted difference scheme for problem (1.1)–(1.2), but only received uniform convergence for \( t \geq M \delta (0 < \delta < 1) \), Hsiao, Jordan[2] also gave a modified Crank-Nicolson-Galerkin scheme for the problem

\[ \varepsilon \frac{\partial u}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left[ a_{ij}(x) \frac{\partial u}{\partial x_j} \right] + c(x) u = f(x,t) \]

but only received uniform convergence for \( t \geq M \delta (0 < \delta < 1) \). In fact, they don’t consider the boundary layer \( t = 0 \). Here, we shall construct difference scheme with non-uniform time step by Bakhvalov[3], and prove the uniform convergence in \( \varepsilon \) for all \( t \in [0, T] \).

II. Estimates of the Solution Derivatives

For the following of this paper we shall assume that \( a \) and \( b \) are independent of \( t \), and \( \psi_0(t) = \psi_1(t) = 0 \).

Theorem 1  Let \( u(x,t) \) be the solution to problem (1.1)–(1.2), if \( Lu < 0 \) and \( u(x,t) > 0 \) in boundary \( \Gamma \), then \( u(x,t) > 0 \) on \( D \).
Proof It's easy to prove it by IL'in [4].

Lemma 1 Let \( u(x,t) \) be the solution to problem (1.1) - (1.3), if \( |f(x,t,e)| \leq N \) on \( \mathcal{D} \), and \( |u(x,t)| \leq m \) in \( \Gamma \), then \( |u(x,t)| \leq \max \{ N/e, m \} \), where \( N \) and \( m \) are arbitrary positive constants independent of \( e \).

Proof Let

\[
  w(x,t) = \max \{ N/e, m \} \pm u(x,t)
\]

then

\[
  Lw = -c \cdot \max \{ N/e, m \} \pm Lu(x,t) \leq -N \pm Lu(x,t) \leq 0
\]

and

\[
  w(x,t) \geq m \pm u(x,t) \geq 0 \quad \text{in} \quad \Gamma
\]

so \( |u(x,t)| \leq \max \{ N/e, m \} \) using Theorem 1.

Theorem 2 Assume that \( u(x,t) \) is the solution to problem (1.1) - (1.3), then

\[
  \frac{\partial u(x,t)}{\partial t} \leq M \left( 1 + e^{-\frac{\partial t}{e}} \right), \quad (x,t) \in \mathcal{D}
\]

\[
  \frac{\partial^2 u(x,t)}{\partial t^2} \leq M \left( 1 + e^{-\frac{\partial t}{2e}} \right), \quad (x,t) \in \mathcal{D}
\]

\[
  \frac{\partial^k u(x,t)}{\partial x^k} \leq M, \quad k = 1, 2, 3, 4, \quad (x,t) \in \mathcal{D}
\]

\[
  \frac{\partial^k u(x,t)}{\partial x^k \partial t} \leq \frac{M}{e}, \quad k = 1, 2, \quad (x,t) \in \mathcal{D}
\]

Proof First, we estimate \( \partial u / \partial t \). Let \( w_1(x,t) = M \left( 1 + e^{-\frac{\partial t}{e}} \right) \pm \partial u / \partial t \), on the side \( t = 0 \) we have \( |\partial u / \partial t| \leq M_1/e \) by (1.1) - (1.2), thus for \( M \) sufficiently large, \( w_1(x,t) \geq 0 \). On the sides \( x = 0 \) and \( x = 1 \) we have \( u = 0 \) and hence \( \partial u / \partial t = 0 \), thus \( w_1(x,t) \geq M_1/e \). Now

\[
  Lw_1 = M \left\{ -c \cdot \left[ 1 + e^{-\frac{\partial t}{e}} \right] - e \cdot e^{-\frac{\partial t}{e}} \cdot \left[ -\frac{\partial f}{\partial t} + u \frac{\partial c}{\partial t} \right] \right\} \leq -M \frac{\partial t}{e} \left[ f / \partial t + u \partial c / \partial t \right]
\]

So for \( M \) sufficiently large, \( Lw_1 \leq 0 \), thus \( |\partial u / \partial t| \leq M \left( 1 + e^{-\frac{\partial t}{e}} \right) \) using Theorem 1.

Then, we shall estimate \( \partial^2 u / \partial t^2 \). Let \( w_2(x,t) = M \left( 1 + e^{-\frac{\partial t}{2e}} \right) \pm \partial^2 u / \partial t^2 \), on the side \( t = 0 \), differentiate (1.1) with respect to \( t \) and let \( t = 0 \), and so we have \( |\partial^2 u / \partial t^2| \leq M_2/e^2 \), thus for \( M \) sufficiently large, \( w_2 \geq 0 \). On the sides \( x = 0 \) and \( x = 1 \), by assumption \( u = 0 \), then \( \partial^2 u / \partial t^2 = 0 \), thus \( w_2 \geq 0 \). Now

\[
  Lw_2 = M \left\{ -c \left[ 1 + e^{-\frac{\partial t}{2e}} \right] - e \cdot e^{-\frac{\partial t}{2e}} \cdot \left[ -\frac{\partial f}{\partial t} + u \frac{\partial c}{\partial t} \right] \right\} \pm L \left( \frac{\partial^2 u}{\partial t^2} \right)
\]