NUMERICAL METHODS FOR PARABOLIC EQUATION WITH A SMALL PARAMETER IN TIME VARIABLE

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(Received Nov. 6, 1990)

Abstract

In this paper, we discuss the parabolic equation with a small parameter on the derivative in time variable. We construct difference scheme on the non-uniform mesh according to Bakhvalov, and prove the one-order uniform convergence of this scheme. Numerical results are presented.

Key words difference scheme, uniform convergence, parabolic type equation, non-uniform mesh

I. Introduction

In this paper we discuss a parabolic equation with a small parameter in time variable,

\[ Lu = a(x,t) \frac{\partial^2 u}{\partial x^2} + b(x,t) \frac{\partial u}{\partial x} - c(x,t) u - \varepsilon \frac{\partial u}{\partial t} = f(x,t,\varepsilon), \quad (x,t) \in D \]

\[ u(x,0) = \varphi(x), \quad u(0,t) = \psi_0(t), \quad u(1,t) = \psi_1(t) \quad (1.2) \]

where \( D = \{ 0 < x < 1, \quad 0 < t \leq T \} \) , and \( a, b, c, f, \varphi, \psi_0 \) and \( \psi_1 \) are sufficiently smooth with \( c(x,t) > \varepsilon > 0 \) (1.3)

on \( D \). Titov\(^{(1)}\) previously constructed the exponentially fitted difference scheme for problem (1.1)–(1.2), but only received uniform convergence for \( t \geq M \delta (0 < \delta < 1) \) , Hsiao, Jordan\(^{(2)}\) also gave a modified Crank-Nicolson-Galerkin scheme for the problem

\[ \varepsilon \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left[ a_{i,j}(x) \frac{\partial u}{\partial x_j} \right] + c(x) u = f(x,t) \]

but only received uniform convergence for \( t \geq M \delta (0 < \delta < 1) \) . In fact, they don’t consider the boundary layer \( t = 0 \). Here, we shall construct difference scheme with non-uniform time step by Bakhvalov [3], and prove the uniform convergence in \( \varepsilon \) for all \( t \in [0, T] \)

II. Estimates of the Solution Derivatives

For the following of this paper we shall assume that \( a \) and \( b \) are independent of \( t \), and \( \psi_0(t) = \psi_1(t) = 0 \).

Theorem 1 Let \( u(x,t) \) be the solution to problem (1.1)–(1.2), if \( Lu < 0 \) and \( u(x,t) > 0 \) in boundary \( \Gamma \), then \( u(x,t) > 0 \) on \( D \).
Proof It's easy to prove it by IL'in [4].

Lemma 1 Let \( u(x,t) \) be the solution to problem (1.1)–(1.3), if \( |f(x,t,\varepsilon)| \leq N \) on \( D \), and \( |u(x,t)| \leq m \) in \( \Gamma \), then \( |u(x,t)| \leq \max\{N/\varepsilon, m\} \), where \( N \) and \( m \) are arbitrary positive constants independent of \( \varepsilon \).

Proof Let

\[
\begin{align*}
\omega(x,t) &= \max\{N/\varepsilon, m\} \pm u(x,t) \\
L\omega &= -c \cdot \max\{N/\varepsilon, m\} \pm Lu(x,t) \leq -N \pm Lu(x,t) \leq 0 \\
\text{and} \\
\omega(x,t) &\geq m \pm u(x,t) \geq 0 \text{ in } \Gamma
\end{align*}
\]

so \( |u(x,t)| \leq \max\{N/\varepsilon, m\} \) using Theorem 1.

Theorem 2 Assume that \( u(x,t) \) is the solution to problem (1.1)–(1.3), then

\[
\begin{align*}
\left| \frac{\partial u(x,t)}{\partial t} \right| &\leq M\left\{ 1 + e^{-1}\exp\left[-\frac{\varepsilon t}{e}\right] \right\}, \quad (x,t) \in \overline{D} \\
\left| \frac{\partial^2 u(x,t)}{\partial t^2} \right| &\leq M\left\{ 1 + e^{-2}\exp\left[-\frac{\varepsilon t}{2e}\right] \right\}, \quad (x,t) \in \overline{D} \\
\left| \frac{\partial^k u(x,t)}{\partial x^k \partial t} \right| &\leq M, \quad k = 1, 2, 3, 4, \quad (x,t) \in \overline{D} \\
\left| \frac{\partial^{k+1} u(x,t)}{\partial x^k \partial t} \right| &\leq \frac{M}{e}, \quad k = 1, 2, \quad (x,t) \in \overline{D}
\end{align*}
\]

Proof First, we estimate \( \partial u/\partial t \). Let \( \omega_1(x,t) = M\{1 + e^{-1}\exp[-\varepsilon t/e]\} \pm \partial u/\partial t \), on the side \( t = 0 \) we have \( |\partial u/\partial t| \leq M_1/e \) by (1.1)–(1.2), thus for \( M \) sufficiently large, \( \omega_1(x,t) > 0 \). On the sides \( x = 0 \) and \( x = 1 \) we have \( u \equiv 0 \) and hence \( \partial u/\partial t = 0 \), thus \( \omega_1(x,t) > M > 0 \). Now

\[
L\omega_1 = M\left\{ -c \cdot \left[ 1 + e^{-1}\exp\left[-\frac{\varepsilon t}{e}\right] \right] - e \cdot e^{-1} \cdot \exp\left[-\frac{\varepsilon t}{e}\right] \right\} \pm L\left( \frac{\partial u}{\partial t} \right)
\]

\[
= \frac{M}{e} \left[ -c - (1 - e\varepsilon) e^{-1}\exp\left[-\frac{\varepsilon t}{e}\right] \right] \pm \frac{\partial f}{\partial t} + u \frac{\partial c}{\partial t}
\]

\[
\leq -M \varepsilon \leq \max\{N/\varepsilon, m\} \pm \partial^2 u/\partial t^2 + u \partial c/\partial t
\]

So for \( M \) sufficiently large, \( L\omega_1 \leq 0 \), thus \( |\partial u/\partial t| \leq M\{1 + e^{-1}\exp[-\varepsilon t/e]\} \) using Theorem 1.

Then, we shall estimate \( \partial^2 u/\partial t^2 \). Let \( \omega_2(x,t) = M\{1 + e^{-2}\exp[-\varepsilon t/2e]\} \pm \partial^2 u/\partial t^2 \), on the side \( t = 0 \), differentiate (1.1) with respect to \( t \) and let \( t = 0 \), and so we have \( |\partial^2 u/\partial t^2| \leq M_2/e^2 \), thus for \( M \) sufficiently large, \( \omega_2 > 0 \). On the sides \( x = 0 \) and \( x = 1 \), by assumption \( u \equiv 0 \), then \( \partial^2 u/\partial t^2 = 0 \), thus \( \omega_2 > 0 \). Now

\[
L\omega_2 = M\left\{ -c \left[ 1 + e^{-2}\exp\left[-\frac{\varepsilon t}{2e}\right] \right] - e \cdot e^{-2} \cdot \exp\left[-\frac{\varepsilon t}{2e}\right] \right\} \pm L\left( \frac{\partial^2 u}{\partial t^2} \right)
\]