DEFORMATION OF STRUCTURE AND SPECTRUM OF EVOLUTION EQUATIONS*

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Abstract

In this paper, we study the general structure of evolution equations of the AKNS eigenvalue problem \( q(x,t), r(x,t) \) with the spectrum varying as

\[
\lambda_i = \sum_{j=-m}^{m} K_j \lambda^{n_j - j} \quad (n_j \leq m_1)
\]

and \( A_1, B_1, C_1 \) are all positive or negative power polynomials of \( \lambda \), where \( q, r \) are not limited with any additional conditions at infinity.

Key words AKNS eigenvalue problem, spectrum deformation, null curvature equation, general structure of evolution equation

Suppose the AKNS eigenvalue problem

\[
\varphi_x = U \varphi; \quad U = \begin{bmatrix} \lambda/2 & q(x,t) \\ r(x,t) - \lambda/2 \end{bmatrix}, \quad \varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}
\]

where \( \psi \) is assumed to satisfy the corresponding evolution equation

\[
\varphi_x = V \varphi, \quad V = \begin{bmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{bmatrix}
\]

When the spectrum varies as

\[
\lambda_i = \sum_{j=1}^{m} K_j \lambda^{m_j - j}
\]

and \( A_1, B_1, C_1 \) are positive power polynomials of \( \lambda \), Li Yi-shen obtained the general structure of evolution equations of \( q, r \).

In this paper, we study the general structure of evolution equation of \( q, r \) with the spectrum varying as

\[
\lambda_i = \sum_{j=0}^{m_1} K_j \lambda^{n_j - j} \quad (n_j \leq m_1)
\]

and \( A_1, B_1, C_1 \) are all positive or negative power polynomials of \( \lambda \), where \( q, r \) are not limited with any additional conditions at infinity, which generalizes the work of [1].

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We know that $U, V$ of (1) and (2) must satisfy the null curvature equation

$$U_t = V_x - [U, V]$$

where

$$[U, V] = UV - VU$$

Denote

$$U = \lambda h_0 + qe_0 + rf_0$$

where

$$h_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad e_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad f_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Suppose

$$V = (A + \varepsilon)h_0 + (B + \delta)e_0 + (C + \theta)f_0$$

according to the null curvature equation, we have

$$\lambda h_0 + qe_0 + rf_0 = (A + \varepsilon + 2B\varepsilon - 2Cq + 2\delta r - 2q\theta)h_0$$

$$+ (B + \delta + Aq - B\lambda + \varepsilon q - \delta \lambda) e_0 + (C + \theta + C\lambda - Ar + \theta \lambda - r\varepsilon) f_0$$

Thus, if

$$\lambda = \varepsilon - 2\theta q + 2\delta r, \quad A + 2B\varepsilon - 2Cq = 0$$

$$q = B\varepsilon + Aq - B\lambda + \delta \lambda - r\varepsilon, \quad r = C + C\lambda - Ar + \theta \lambda - r\varepsilon$$

then (3) is satisfied.

Suppose

$$A = \sum_{j=0}^{m} a_j \lambda^{n_j - j}, \quad B = \sum_{j=0}^{m} b_j \lambda^{n_j - j}, \quad C = \sum_{j=0}^{m} c_j \lambda^{n_j - j}$$

$$\lambda = \sum_{j=0}^{n_1} \lambda^{n_j - j}$$

$$\varepsilon = \sum_{j=0}^{m} \varepsilon_j \lambda^{n_j - j}, \quad \delta = \sum_{j=0}^{m} \delta_j \lambda^{n_j - j}, \quad \theta = \sum_{j=0}^{n_1} \theta_j \lambda^{n_j - j}$$

where $n < m, n_j < m_j$.

Now we discuss two cases of $n < m, n_j < m_j$ and $n = m, n_j < m_j$. We can study the case of $n = m, n_j = m_j$ in the same way.

(1) $n < m, n_j < m_j$

Letting (7), (8), (9) substitute (6), we have

$$b_0 = 0, \quad b_{j+1} = a_j g - b_j + 1 = 0, \quad b_m + a_m q = 0 \quad (j = 0, 1, \ldots, m - 1; \ j \neq n)$$

$$c_0 = 0, \quad c_{j+1} = a_j r - c_j + 1 = 0, \quad c_m - a_m r = 0$$

$$a_{j+1} = 2c_j g - 2b_j r \quad (j = 0, 1, \ldots, m)$$

$$\delta_0 = 0, \quad \delta_{j+1} + \varepsilon_j g = 0, \quad \delta_{m+1} + \varepsilon_m q = 0 \quad (j = 0, 1, \ldots, m - 1; \ j \neq n_1)$$

$$\theta_0 = 0, \quad \theta_{j+1} - \varepsilon_j r + \theta_j + 1 = 0, \quad \theta_{m+1} - \varepsilon_m r = 0$$