BEST APPROXIMATION THEOREM FOR SET-VALUED MAPPINGS
WITHOUT CONVEX VALUES AND CONTINUITY*

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Abstract

In this paper, a new concept of weakly convex graph for set-valued mappings is introduced and studied. By using the concept, some new coincidence, the best approximation and fixed point theorems are obtained.

Key words best approximation, coincidence, fixed point, topological vector space

I. Introduction

In 1969, Ky Fan proved the following well-known results on best approximation.

**Theorem 1.1** Let X be a nonempty compact convex subset of a locally convex topological vector space E and \( f: X \rightarrow E \) be a continuous mapping. Then either \( f \) has a fixed point in \( X \), or there exist a point \( y_0 \in X \) and a continuous seminorm \( p \) on \( E \) such that

\[
0 < p(y_0 - f(y_0)) = \min \{ p(x - f(y_0)) : x \in X \}
\]

Since then many authors have generalized the above result to set-valued mappings with compact convex values, e. g., see [2~8].

In 1987, Hu generalized Fan's best approximation theorem to set-valued mapping as follows.

**Theorem 1.2** Let \( X \) be a nonempty compact convex subset of a locally convex topological vector space \( E \) and let \( T: X \rightarrow 2^E \) be an upper semicontinuous set-valued mapping such that \( T(x) \) is compact and convex for each \( x \in X \). Then either \( T \) has a fixed point, or there exist \( x_0 \in X \), \( u_0 \in T(x_0) \), and a continuous seminorm \( p \) on \( E \) such that

\[
0 < p(x_0 - u_0) \leq p(x - u_0) \quad (\forall x \in X)
\]

Recently, Ding-Tan and Ding-Tarafdar generalized the Hu's result to noncompact setting and to a pairing of mappings under weaker assumptions.

In this paper, we introduce a new notion of weakly convex graph for set-valued mappings in topological vector spaces. By using the notion and different argument method, some new coincidence, best approximation and fixed point theorems for set-valued mappings without compact convex values and continuity are proved in topological vector spaces.

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II. Preliminaries

Let $X$ be a nonempty convex subset of a topological vector $E$ and $Y$ be a nonempty subset of $E$. We shall denote by $2^Y$ the family of all nonempty subsets of $Y$ and by $\mathcal{P}$ the family of all continuous seminorm on $E$. For a subset $A$ of $E$, we shall denote by $\text{co}(A)$ the convex hull of $A$. Let $T: X \to 2^Y$ be a set-valued mapping. The graph of $T$ is the set $G_r(T) = \{(x,y) \in X \times Y : y \in T(x)\}$. $T$ is said to have compact graph if $G_r(T)$ is compact in $X \times Y$. $T$ is said to have weakly convex graph if for each finite set $\{x_1, x_2, \ldots, x_n\} \subseteq X$, there exists $y_i \in T(x_i), (i = 1, 2, \ldots, n)$, such that

$$\text{co}\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\} \subseteq G_r(T)$$

(2.1)

Let $\sigma = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, i = 1, 2, \ldots, n\}$ be a $(n-1)$-dimensional simplex. Then the relation (2.1) is equivalent to

$$\sum_{i=1}^{n} \lambda_i y_i \in T\left( \sum_{i=1}^{n} \lambda_i x_i \right) \quad (\forall (\lambda_1, \ldots, \lambda_n) \in \sigma)$$

(2.2)

It is clear that if either the graph $G_r(T)$ of $T$ is convex, or $\bigcap \{T(x) : x \in X\} \neq \emptyset$, then $T$ has weakly convex graph.

Lemma 2.1: Let $T: X \to 2^Y$ be a set-valued mappings with weakly convex graph and $X_0$ be a nonempty convex subset of $X$. Then the restriction of $T$ on $X_0$, $T|_{X_0}: X_0 \to 2^Y$ also has weakly convex graph.

Proof: For each finite set $\{x_1, \ldots, x_n\} \subseteq X_0 \subseteq X$, by the weak convexity of $G_r(T)$, there exist $y_i \in T(x_i), (i = 1, 2, \ldots, n)$ such that

$$\sum_{i=1}^{n} \lambda_i y_i \in T\left( \sum_{i=1}^{n} \lambda_i x_i \right) \quad (\forall (\lambda_1, \ldots, \lambda_n) \in \sigma)$$

Since $X_0$ is convex, we have $\sum_{i=1}^{n} \lambda_i x_i \in X_0$ and hence $T\left( \sum_{i=1}^{n} \lambda_i x_i \right) = T|_{X_0}\left( \sum_{i=1}^{n} \lambda_i x_i \right)$. It follows that

$$\sum_{i=1}^{n} \lambda_i y_i \in T|_{X_0}\left( \sum_{i=1}^{n} \lambda_i x_i \right) \quad (\forall (\lambda_1, \ldots, \lambda_n) \in \sigma)$$

This shows that $T|_{X_0}$ has weakly convex graph.

The following examples show that a mapping $T: X \to 2^Y$ with weakly convex graph may not be upper semicontinuous and convex-valued.

Example 2.1: Let $Y = E = \mathbb{R}^2$ and $X = \{(x,0) : x \in \mathbb{R}\}$. Define $T: X \to 2^Y$ by

$$T((x,0)) = \{(x,y) : y \in \mathbb{R}\} \quad (\forall (x,0) \in X)$$

It is easy to see that $T$ is not upper semicontinuous at $(0, 0)$. In fact, the set $N = \{(x, y) : |y| < \frac{1}{x}\}$ is an open neighborhood of $T((0,0))$, but for any $x \neq 0$, $T((x,0))$ cannot be contained in $N$. Now we prove that $T$ has weakly convex graph. Indeed, for each finite set $\{(x_1,0), \ldots, (x_n,0)\} \subseteq X$, let $y \in \mathbb{R}$ be an any fixed point, then we have $(x_i, y) \in T((x_i,y)) (i = 1, \ldots, n)$ and $\text{co}\{(x_0,0), \ldots, (x_n,0)\} \subseteq G_r(T)$. Therefore $G_r(T)$ is weakly convex.

Example 2.2: Let $E = \mathbb{R}, X = [0,2], Y = [4,6]$. Define $T: X \to 2^Y$ by