CHAOTIC BEHAVIOR IN THE HELLEMAN MAPPING*

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Abstract

In this paper, we establish the analytical conditions for the Helleman mapping in which the Smale horseshoe appears. Then we use it to deduce the chaotic criterion of Henon maps.

Helleman[1] considered the "standard form" \( \varphi : \)
\[
\begin{align*}
x_{n+1} &= 2Cx_n + 2C^2x_n^2 - y_n \\ y_{n+1} &= Bx_n
\end{align*}
\]

Zisook[2] used it to examine the new speciality for 2-dimensional nonlinear mapping.

For the general case, in the present paper, we can determine a square basin \( Q \) in the \((x,y)\) plane. If the \( 2R, B, C \) satisfy the relationship:
\[
\begin{align*}
0 \leq B &\leq 1, \quad C > 2(1 + B), \\
0 < B &< 1, \quad C < - (1 + B), \\
(1 + B - C) &< R < A/(1 + B), \\
-1 < B &< 0, \quad C > A_1 \text{ or } C < A_2, \\
A_3 &< R < A/(1 - B)
\end{align*}
\]

Here
\[
\begin{align*}
A &= C^2 - BC - C \\
A_1 &= [(1 + B) + \sqrt{(1 + B)^2 + 8(1 - B)^2}]/2 \\
A_2 &= [(1 + B) - \sqrt{(1 + B)^2 + 8(1 - B)^2}]/2 \\
A_3 &= [(1 - B) + \sqrt{(1 - B)^2 + 4A}]/2
\end{align*}
\]
in \( Q \). Then \( \varphi \) possesses a shift automorphism \( \sigma \) of \( S \) as a subsystem, that is to say, there is a homeomorphism \( \tau : S \rightarrow A \subset Q \) with \( \varphi \circ \tau = \tau \circ \sigma \), where \( S \) is a set of double infinite sequences.

Then we use it to deduce the chaotic criterion for the Henon mapping.

For convenience, let
\[
\begin{align*}
2Cx_n &= -(x + C), \\
2Cy_n &= -(y + C), \\
2Cx_{n+1} &= -(u + C), \\
2Cy_{n+1} &= -(v + C)
\end{align*}
\]

We can express \( \varphi \) as the equivalent form:
\[
\varphi(x, y) = (- x^2 - By + A, x)
\]

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Take the fundamental basin $Q$:

$$-R \leq x \leq R$$

$$-R \leq y \leq R$$

We have

**Lemma 1** \( \varphi(Q) \cap Q \) is a set composed of two horizontal strip regions \( U_1 \) and \( U_2 \), their bound areas are parabolas:

$$u = -v^2 + A - BR$$  \hspace{1cm} (I)

$$u = -v^2 + A + BR$$  \hspace{1cm} (II)

and \( u = -R, u = +R \) (Fig. 1).

**Proof** The image of \( y = \pm R \) are parabolas (I) and (II).

For \( C < 0 \) or \( C > (1+B) \), if \( 0 \leq B \leq 1, R < A/(1+B) \) or \(-1 \leq B < 0, R < A/(1-B) \). Then the vertex of parabolas (I) and (II) are all on the right to line \( u = R \).

We can compute the intersection of (I), (II) and \( u = \pm R \) are

$$N_1: (-R, \sqrt{A+(1-B)R})$$

$$N_2: (R, \sqrt{A-(1+B)R})$$

$$N_3: (R, -\sqrt{A-(1+B)R})$$

$$N_4: (-R, -\sqrt{A+(1-B)R})$$

and

$$N_5: (-R, \sqrt{A+(1+B)R})$$

$$N_6: (R, \sqrt{A-(1-B)R})$$

$$N_7: (R, -\sqrt{A-(1-B)R})$$

$$N_8: (-R, -\sqrt{A+(1-B)R})$$

**Lemma 2** The preimages of \( U_1 \) and \( U_2 \) are vertical regions \( V_1 \) and \( V_2 \) in \( Q \), the bound areas of which are (III), (IV) and \( y = \pm R \) (Fig. 2).