NOTE

BOUNDS FOR GENERALIZED DREITLEIN–SMOES MODELS

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The analysis of a previous paper obtaining bounds on the total population number of species (chemical or biological) described by the recently proposed Dreitlein–Smoes model of oscillatory kinetic systems, including diffusion, is extended to generalized models of the Dreitlein–Smoes type, describing a system of $S$ components with $S > 2$. The results for such generalized models are analogous to those of the previous case. It is found that the effects of diffusion serve to restrict the region in the concentration space available to limit-cycle type oscillations.

1. Introduction. The recently proposed model of Dreitlein and Smoes (Smoes and Dreitlein, 1973; Dreitlein and Smoes, 1974) for oscillatory kinetic systems has been analyzed in a previous paper (Fizell and Rubin, 1976) for bounds on the total population of the interacting species (chemical or biological) where a full account of diffusion of the oscillating species is taken. It was found that the effect of diffusion is to limit the maximum possible variations in limit-cycle type behavior. In particular, it was shown that in some cases such limit-cycle behavior is precluded.

In this paper, the analysis is extended to certain types of generalized Dreitlein–Smoes models (Dreitlein and Smoes, 1974) where again diffusion will be included. After certain transformations on the original equations, a form is arrived at that allows the immediate application of the preceding (Fizell and Rubin, 1976) results.
2. Generalized Models. We consider the system (Dreitlein and Smoes, 1974)

\[
\frac{\partial X_i}{\partial t} = D \nabla^2 X_i + \sum_j \Omega_{ij} X_j + [E - l(X)]X_i
\]  

(1)

in a closed, bounded two- or three-dimensional region \( R \) for \( i = 1, 2, \ldots, S \) where the \( X_i \) are the components of the \( S \)-dimensional vector \( X \) of relative concentrations, \( D \) is the (positive) diffusion coefficient, \( \Omega \) is an \( S \times S \) matrix, \( E \) is a constant parameter, and \( l(X) \) is a nonlinear scalar function of the \( X_i \).

We will impose on \( l \) the conditions

\[
\sum_{i=1}^{S} \frac{\partial l(X)}{\partial X_i} X_i = n l(X) \quad \text{(2a)}
\]

for some \( n \) and

\[
\sum_{j,k=1}^{S} \frac{\partial l(X)}{\partial X_j} \Omega_{jk} X_k = 0. \quad \text{(2b)}
\]

Note that (2a) says that \( l \) is a homogeneous function of the \( X_i \) of order \( n \). Equation (1) with (2a, b) implies that \( l(X) \) will satisfy the equation

\[
\frac{\partial l}{\partial t} = D \sum_{i=1}^{S} \frac{\partial l}{\partial X_i} \nabla^2 X_i + n(E - l) l. \quad \text{(3)}
\]

We will consider the case where \( l(X) \) is a homogeneous quadratic polynomial. Then \( l(X) \) may be written in the form

\[
l(X) = \sum_{i,j=1}^{S} X_i \alpha_{ij} X_j, \quad \text{(4)}
\]

where \( \alpha_{ij} = \alpha_{ji} \). Thus, (3) becomes

\[
\frac{\partial l}{\partial t} = 2D \sum_{i,j=1}^{S} \alpha_{ij} X_j \nabla^2 X_i + 2(E - l) l. \quad \text{(5)}
\]

Now consider the \( S \times S \) orthogonal matrix \( U \) that diagonalizes the \( \alpha_{ij} \) matrix:

\[
\sum_{k,l=1}^{S} (U^T)_{kl} \alpha_{kl} U_{lj} = \overline{\alpha}_{lj} \delta_{lj}. \quad \text{(6)}
\]

Then with the vector \( Y \) defined by

\[
Y_i = \sum_j (U^T)_{lj} X_j, \quad \text{(7)}
\]

we have

\[
l(X) = \sum_i \overline{\alpha}_i Y_i^2. \quad \text{(8)}
\]