A STUDY OF THE STATIC AND GLOBAL BIFURCATIONS FOR DUFFING EQUATION*

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Abstract: In this paper, the static and global bifurcations of the forced Duffing equation have been studied by means of the averaged system. Bifurcation condition has been obtained in the whole parametric space. The change of the phase plane structure has been investigated.

Key words: forced Duffing equation; averaged system; bifurcation

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Introduction

As is known to us, Duffing equation is one of the most widely used nonlinear oscillator being of plentiful dynamical behaviors, which are the interesting subject up to now in the world[1-3].

In this paper, both the static and the global bifurcations of the forced Duffing equation have been explored by means of the averaged system. The bifurcation condition has been obtained for the system in the whole parametric space. The change of the phase plane structure and the nonlinear dynamical behavior have been investigated. The results obtained in this paper extended the results both in [2] and [3].

1 Averaged System

Consider Duffing equation
\[ x + 2n\dot{x} + \omega_0^2 x + a_0 x^3 = F \cos \omega t, \] (1)
where \( \omega_0 \) is the nature frequency, \( n \) is the damping parameter, \( a_0 \) is the nonlinear parameter, \( F \) and \( \omega \) are amplitude and frequency of the external influence respectively. Parameters \( \omega_0^2 - \omega^2 = \Omega \), \( n \), \( a_0 \) and \( F \) are small quantity in the degree of \( \varepsilon \). Without confusion, we will omit the symbol \( \varepsilon \) in the following discussion.

Let
\[ \begin{cases} x = u \cos \omega t + v \sin \omega t, \\ \dot{x} = -u \omega \sin \omega t + v \omega \cos \omega t, \end{cases} \] (2)

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we have
\[
\begin{align*}
    \begin{cases}
        u'\cos \omega t + v'\sin \omega t = 0, \\
        - u'\omega \sin \omega t + v'\omega \cos \omega t = f(u, v, t),
    \end{cases}
\end{align*}
\]  
(3)

where
\[
f(u, v, t) = F\cos \omega t - \alpha_0(u\cos \omega t + v\sin \omega t)^3 - 2\pi(- u\omega \sin \omega t + v\omega \cos \omega t) - \Omega(u\cos \omega t + v\sin \omega t),
\]

which implies that
\[
\begin{align*}
    u' &= - \frac{1}{\omega} f(u, v, t)\sin \omega t, \\
    v' &= \frac{1}{\omega} f(u, v, t)\cos \omega t.
\end{align*}
\]  
(4)

Averaging system (4) over a period \([0, 2\pi]\) follows that
\[
\begin{align*}
    \begin{cases}
        u' &= - nu + 2\sigma_1 v + 8\sigma_2 v, \\
        v' &= - 2\sigma_1 u - nv - 8\sigma_2 u + \frac{F}{2\omega}.
    \end{cases}
\end{align*}
\]  
(5)

Let \( r = \sqrt{u^2 + v^2} \), \( \varphi = \arctan(v/u) \) follows the amplitude and phase angle equation
\[
\begin{align*}
    \begin{cases}
        r' &= nr + \frac{F}{2\omega} \sin \varphi, \\
        r\varphi' &= - \frac{\Omega}{2\omega} - \frac{3\sigma_0}{8\omega}(u^2 + v^2) + \frac{F}{2\omega} \cos \varphi,
    \end{cases}
\end{align*}
\]  
(6)

which leads to the responding or amplitude equation given by
\[
\frac{F^2}{4\omega^2} = n^2 r^2 + r^2 \left[ \frac{\Omega}{2\omega} + \frac{3\sigma_0}{8\omega} \right]^2.
\]  
(7)

Fig. 1 Amplitude for Duffing equation

Fig. 2 Stable and unstable solutions

It can be seen from Fig. 1 and Fig. 2 that the number of solutions changes from one to two, from two to three, and then from three to two and one as parameter \( \omega \) varies.