SEMI-LINEAR STOCHASTIC MODELS:
COMPUTATION OF MOMENTS AND
CONDITIONAL MOMENTS

E. PARDOUX
U.E.R. de Mathématiques,
Université de Provence,
3, place V. Hugo,
13331 Marseille Cedex 3, France

In this article we show how linearity with respect to the output of a stochastic dynamic model can be exploited in order to simplify the computation of moments or conditional moments. The results are presented for two examples, one of which includes delays. This feature is often encountered in biological models.

Introduction. A stochastic dynamical model, e.g. for a biological phenomenon, can be viewed as an input–output map, where both the input and output are stochastic processes. In order to make any relevant conclusion about the output one needs information about the law of the input process (given by the assumptions and/or by a statistical estimation procedure from the observed data). It may be of interest to compute such quantities as the moments of the output. In other situations (e.g. in a medical application, the non-observable output being the evolution of a disease) a partial observation of the input or output is available, and one wants to compute the conditional expectation of the output, given the observation.

In this article we give results which permit a simplification of such computations in the case where the model is partially linear. The models considered here include possible delays.

The results in Section 1 have been obtained jointly with R. Bouc, and appear in a more general form in Bouc and Pardoux (1981). The results in Section 2 appear with complete proofs in Pardoux (1982).

1. Computation of Moments.

1(a). Statement of the problem. Let $X_t$ be the solution of the Itô stochastic differential equation in dimension $n$:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \ X_0 = \text{given}$$

r.v., whose law has density $p_0(x)$. Then $X_t$ is a Markov diffusion process, with generator:
\[ L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i}. \]

\( X_t \) is the input of the following differential equations:

\[ \frac{dU_t}{dt} = A(X_t)U_t + g(X_t), \quad U_0 \in \mathbb{R}^m \quad (1.1) \]

\[ \frac{dU_t}{dt} = \int_{0}^{t} G(t, s, X_p)U_p ds, \quad U_0 \in \mathbb{R}, \quad (1.2) \]

where \( A(\cdot), g(\cdot) \) and \( G(\cdot, \cdot, \cdot) \) are measurable and bounded functions defined on \( \mathbb{R}^m, \mathbb{R}^m \) and \( \mathbb{R}_+ \times \mathbb{R}^m \) respectively, with values in \( \mathcal{L}(\mathbb{R}^n) \), \( \mathbb{R}^n \) and \( \mathbb{R} \) respectively.

We want to give equations for the computation of \( E(U_t) \). In case of model (1.1) the result below can be extended to higher-order moments. In case of model (1.2) the result is specific to the first moment.

**1(b). The Kolmogorov backward and forward equation.** Let us first concentrate upon the process \( X_t \) alone. Consider the Kolmogorov backward equation attached to \( X_t \):

\[ \frac{\partial}{\partial s} v(s, x) + L v(s, x) = 0; s \leq t, x \in \mathbb{R}^n \]

\[ v(t, x) = f(x), f \in C_b(\mathbb{R}^n) \quad (1.3) \]

It is easy to show that \( v \) is given by:

\[ v(s, x) = E[f(X_t)/X_s = x]. \quad (1.4) \]

Indeed, if we ignore the regularity of the solution \( v \) of (1.3) we may apply the Itô formula to the process \( \psi = v(\theta, X_\theta) \), which yields

\[ \psi_t - \psi_s = \int_{s}^{t} \left( v'_\theta + Lv \right) (\theta, X_\theta) d\theta + \int_{s}^{t} \nabla v \theta (\theta, X_\theta) dW_\theta. \]

Using (1.3) we get

\[ E[\psi_t - \psi_s/X_s = x] = 0, \]

which proves (1.4).

\[ \dagger \text{A rigorous proof needs a regularizing procedure.} \]