DETERMINATION OF TIME-CONSTANTS IN CABLES OF FINITE LENGTH

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Current flow in cylindrical nerve and muscle fibre has been analysed in terms of a mathematical model leading to a linear partial differential equation for the voltage as a function of both position and time. In the case of a one-dimensional cable subject to a step input of current, the solution will consist of a steady-state behaviour preceded by an initial transient. The electrical properties of the fibre or cable itself determine a length-constant, $\lambda$, which can be determined experimentally from the steady-state response, and a time-constant, $\tau$, which must be found from the initial transient.

When the cable is infinite and when there is a single input electrode, an exact solution can be produced which enables ready determination of the time-constant $\tau$.

Two complications arise in experimental practice, however. In the first place, the fibre has finite length, and in the second, two spatially separated stimulation electrodes are often required. We thus analyse a more complicated and more general situation. The linearity of the membrane properties, however, allows the solution to the more general case to be built up by superposition of solutions from the simpler case (equivalent to the classical method of images). We also approximate the Hodgkin and Rushton solution by asymptotic formulae in order to allow more tractable expressions for the exact solution.

We are thus able to give a method for the ready evaluation of the time constant $\tau$ under more general conditions.

1. Introduction. The governing equation for membrane potential change in a one-dimensional cable may be written in a standard (normalized) form (Jack et al., 1975, p. 28) as:

$$\frac{\partial^2 V}{\partial X^2} - \frac{\partial V}{\partial T} - V = 0$$

where $V$ is the change in membrane potential produced by the intracellular application of current;

$X$ measures in normalized units the distance between the stimulating electrode and the point at which $V$ is measured;

and $T$ measures the time in normalized units.
$X, T$ are related to the conventional measures, $x, t$, respectively, of distance and of time by the equations:

$$x = \lambda X, \quad t = \tau T$$

(2)

where $\lambda, \tau$ are referred to as the space- and time-constants, respectively. These quantities relate in their turn to the resistance and capacitance characteristics of the membrane. The experimental determination of $\lambda, \tau$ is thus a matter of some importance in the investigation of those characteristics.

The standard approach envisages an infinite cable, stimulated at $X=0$ by application of a step input of current time $T=0$. The resulting solution may then be written:

$$V = e^{-x} W(X, T)$$

(3)

where:

$$W(X, T) = \frac{1}{2} \left\{ \text{erfc} \left( \frac{X}{2\sqrt{T}} - \sqrt{T} \right) - e^{2x} \text{erfc} \left( \frac{X}{2\sqrt{T}} + \sqrt{T} \right) \right\}$$

(4)

$\text{erfc} z$ being the complementary error function (Abramowitz and Stegun, 1965; p. 297).

It is readily seen that $W(X, T)$ is a monotonic increasing function of $T$, whatever the value of $X$, that $W(X, 0) = 0$ and that $W(X, \infty) = 1$. See Jack et al. (1975, pp. 33–35) for graphs showing the dependence of $W(X, T)$ on $X, T$.

Since the steady state value of $V$ is proportional to $e^{-x}$, the spatial decay of the steady-state solution allows a ready determination of the space-constant $\lambda$. To find $\tau$, however, requires analysis of some suitable feature of the transient behaviour.

The standard technique consists of showing that the value:

$$W(X, T) = \frac{1}{2}$$

(5)

is achieved for a value $T_N$ of $T$, where, to a good approximation:

$$T_N = \frac{1}{2} X + \frac{1}{4}.$$  

(6)

Thus, if, for several values of $X$, the time to 50% of steady-state value is determined, equation (5) allows for determination of the time-constant $\tau$. This relation was first deduced by Hodgkin and Rushton (1946), who note a priority claim by Cremer.

In many areas of practical concern, however, two complications arise. In the first place, the fibre is necessarily of finite length, and in the second, experimental practice requires two stimulation electrodes, which must, together with the recording electrode, fit within that finite length. It is possible