THE EXTENSION OF TWO-DIMENSIONAL CABLE THEORY TO ARTERIES AND ARTERIOLES

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Electrical polarization of an artery or an arteriole may be modeled by the use of equations developed for two-dimensional cable theory. Two special cases have previously been solved: those corresponding to the case in which the radius is either zero (one-dimensional cable theory) or infinite. This paper presents the general solution.

1. Introduction. The effects of electrical polarization of excitable cells have been widely studied both experimentally and theoretically. This research is well summarized by Jack et al. (1975) and so will not be recalled in detail here, except for its most salient features.

The most developed aspect of the theory is linear cable theory, in which a one-dimensional system is excited and effects on transmembrane voltage are measured. To model this system, the linear system is replaced by an equivalent electrical circuit, whose response is then described mathematically.

Two studies extend this work to the simplest possible two-dimensional case: that of a flat sheet of cells, infinite in extent and polarized in the immediate neighbourhood of a single point. Jack et al. (1975) employed a Laplace Transform technique to obtain an explicit solution for the transmembrane voltage as an infinite series of incomplete gamma functions. Shiba and Kanno (1971) on the other hand applied a Green’s function technique and found the solution as an infinite integral involving a Bessel function as kernel.

Neild (1983) studied the excitation of arterioles and small arteries and thus was led to consider the question of solving the case illustrated in Fig. 1. Mathematically, this may be seen to generalize the two previous cases. The linear case appears as the limit when the radius of the cylinder tends to zero, while the other is the corresponding limit as the radius tends to infinity.

This paper presents the solution of the general case. That solution appears as a double series of incomplete gamma functions, whose numerical evaluation presents some technical difficulties. We here present the formal solutions. A subsequent paper will deal with the numerical questions.
2. The Mathematical Problem. The essentially three-dimensional geometry of Fig. 1 becomes two-dimensional in the practical case $\Delta \ll a$. We assume this approximation as, in cases of interest, $a$ never becomes less than $30\Delta$, and is usually larger than this.

Then, in the notation of Shiba and Kanno (1971), the governing equation is

$$\frac{\Delta}{\rho_{xx}} \frac{\partial^2 V}{\partial x^2} + \frac{\Delta}{\rho_{yy}} \frac{\partial^2 V}{\partial y^2} + 4\pi \phi(x, y)f(t) = \frac{2V}{R_m} + 2C_m \frac{\partial V}{\partial t}, \quad (1)$$

where

$V$ is the membrane potential change produced along the cytoplasmic layer by the intracellular application of current;

$\rho_{xx}$ is the resistivity in the direction of the x-axis;

$\rho_{yy}$ is the resistivity in the direction of the y-axis;

$\Delta$ is the thickness of the cytoplasmic layer;

$1/R_m$ is the effective conductance per unit area of the surface membrane (i.e. leakage conductance);

$C_m$ is the effective capacitance per unit area of the surface membrane;

$4\pi \phi(x, y)f(t)$ is the current injected per unit area over an area $S_0$, and such that

$$4\pi \int \int_{S_0} \phi(x, y) \, dx \, dy = 1, \quad (2)$$

and $\phi(x, y) = 0$ outside $S_0$;

$x, y$ are position coordinates; and

$t$ is the time.

Following Shiba and Kanno, set