STOCHASTIC MODEL OF POPULATION GROWTH AND SPREAD

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A stochastic model for the population regulated by logistic growth and spreading in a given region of two- or three-dimensional space has been introduced. For many-species population the interactions among the species have also been incorporated in this model. From the random variables that describe stochastic processes of a Wiener type the space-dependent random population densities have been formed and shown to satisfy the Langevin equations. The Fokker–Planck equation corresponding to these Langevin equations has been approximately solved for the transition probability of the population spreading and it has been found that such approximate expressions of the transition probability depend on the solutions of the deterministic equations of the diffusion model with logistic growth and interaction. Also, the stationary or equilibrium solutions of the Fokker–Planck equation together with the special discussion on the pattern of single-species population spreading have been made.

1. Introduction. The study of population growth and spreading in a region of space leads, in general, to the deterministic equations of reaction–diffusion type. Such types of phenomenological equations occur in various other chemical and biological situations. Stability and bifurcation analyses have been made by several authors (Auchmuty and Nicolis, 1975; Herschkowitz-Kaufmann, 1975; Brown and Eilbeck, 1982). The extrinsic randomness has been introduced in the evolution equations of reaction–diffusion type by the perturbation of all or some of the parameters of the model with white noise. In this manner, the stochastic partial differential equations arise from those deterministic equations. Investigations of such equations, in particular, the existence results for certain classes of problems have been made recently by Da Prato (1983) and Da Prato et al. (1979, 1982).

The aim of the present paper is to introduce a stochastic population model for the population spreading with logistic growth for single-species and also for several-species populations with the interaction among the species. In this model, the random population density dependent on space points has been constructed from the random variables describing stochastic
processes of the Wiener type. The Langevin equations are being obeyed by this population density and the Fokker–Planck equation for the transition probability has been constructed corresponding to these Langevin equations. The approximate method of solutions for the transition probability of the population spreading has been given and shown to depend on the deterministic equations of the diffusion model with logistic growth and interactions among the species. Also, we have discussed the stationary or equilibrium solutions for the single-species population as well as for many-species population. As an illustration, the spatial patterns of a single-species population with spreading and logistic growth for such cases, have been calculated.

We begin, in Section 2, with the introduction of the present model and its governing equations for the single- and two-species populations, in particular. In Section 3, the Fokker–Planck equation for the transition probability has been formed from the Langevin equations and in Section 4 the equilibrium or stationary solutions of this equation have been given, together with the spatial patterns of single-species population. In the subsequent section, the approximate method of solutions for the transition probability has been furnished.

2. The Stochastic Model of Population Spread. Let \( X_i(x,t) \) be the population density of the \( i \)th species \((i=1,2,\ldots,n)\) at time \( t \) at the point \( x \), where \( x \) is a point of \( \mathbb{R}^n \) with \( n=2 \) or \( 3 \). Let \( \zeta_i=(\zeta_{i1},\zeta_{i2},\ldots,\zeta_{im},\ldots) \) be the random variables describing the Wiener-type stochastic processes as functions of time \( t \) and obey the Langevin equations

\[
\frac{d\zeta_{il}(t)}{dt} = K_{il}(\zeta_i,t) + \eta_{il}(t) \quad (i=1,2,\ldots,n)
\]

where \( K_i=(K_{i1},K_{i2},\ldots) \) and \( \eta_i=(\eta_{i1},\ldots,\eta_{im},\ldots) \) stand for systematic and random 'forces' to derive the stochastic processes, respectively. The Langevin equations (1) are supplemented by the statistical property:

\[
\langle \eta_{il}(t) \rangle = 0
\]

\[
\langle \eta_{il}(t) \eta_{lm}(t') \rangle = 2\delta_{im}\delta(t-t')
\]

for random forces, where \( <> \) represent the averaging over \( \eta_i \) with Gaussian probability distribution.

Now, let \( \{f_j(x)\} \) be a complete orthonormal set depending on the coordinates \( x \) of \( \mathbb{R}^n \) \((n=2 \text{ or } 3)\), and we connect the random population densities \( X_i(x,t) \) with the random variables \( \zeta_i \) through