NON-LINEAR COMPARTMENTAL SYSTEMS:
EXTENSIONS OF S. R. BERNARD'S URN MODEL*

MARY G. LEITNAKER
Department of Statistics,
The University of Tennessee,
Knoxville, TN 37996, U.S.A.

PETER PURDUE
Department of Statistics,
University of Kentucky,
Lexington, KY 40506, U.S.A.

One of the limitations of stochastic, linear compartmental systems is the small degree of variability in the contents of compartments. S. R. Bernard's (1981) urn model (S. R. Bernard et al., Bull. math. Biol. 43, 33-45.) which allows for bulk arrivals and departures from a one-compartment system, was suggested as a way of increasing content variability. In this paper, we show how the probability distribution of the contents of an urn model may be simply derived by studying an appropriate set of exchangeable random variables. In addition, we show how further increases in variability may be modeled by allowing the size of arrivals and departures to be random.

1. Introduction. Stochastic compartmental models have typically been constructed by describing the movement of a single particle through a compartmental system. Thakur et al. (1972) consider a one-compartment system where movement through the compartment is described by:

\[ \mu \Delta t + o(\Delta t) = P(\text{a given particle departs in } (t, t + \Delta t)), \quad \mu \geq 0 \]
\[ f(t) \Delta t + o(\Delta t) = P(\text{a given particle enters in } (t, t + \Delta t)), \quad f(t) \geq 0. \]

Purdue (1974) considers a more general system for which entry to the process is described by a nonhomogeneous Poisson Process with parameter \( m(t) \) and the departure process by a general distribution \( F(x, t) \), where \( 1 - F(x, t) \) is the probability that a particle which enters at time \( x > 0 \) is present in the compartment at time \( t \).

One of the consequences of modeling a one-compartment system by considering the behavior of single particles is the small degree of variability present in the model when modeling the movement of a large number of particles. In the models considered by Thakur and Purdue, the coefficient

* Supported by NSF Grant No. MCS 8102215-01.
of variation for $X(t)$, the number of particles present in the compartment at time $t$, is smaller than $(E[X(t)])^{-1/2}$ where $X(0) = N$ and there is no input after time $t = 0$.

Often experimental data exhibits a larger degree of variation than would be accounted for by the single-particle transfer behavior of the above models. Bernard (1977) has suggested that in such cases a more appropriate model may be an urn model for which transfers occur in batches. This model can be described by an urn or compartment which initially contains $b$ black balls and $w$ white balls. The urn undergoes changes in content by what Bernard terms reinforcement–depletion (R–D) cycles. This cycle consists of reinforcing the black balls by adding $r$ black balls, then mixing the balls in the urn and finally depleting the contents of the urn by randomly selecting $r$ balls to be withdrawn from the total of $b + w + r$ balls in the urn. Thus the white balls are undergoing dilution. These R–D cycles are repeated and our interest centers on $W(m)$, the number of white balls present after $m$ R–D cycles.

Bernard proposed this model as a method of describing the number of radioactive atoms present in a ‘compartment’ which began with $w$ radioactive atoms and is losing these atoms over time. In his paper he finds a recursive expression for $P_k(m) = P(W(m) = k)$. In a subsequent paper, Shenton (1981) finds a closed form expression for $P_k(m)$ by using a somewhat lengthy generating function argument. In Section 2 we will show how this expression can be derived by constructing an appropriate set of exchangeable, indicator random variables.

The idea of using such random variables is suggested by Purdue (1981), who uses them to find the expected value and variance of $W(m)$. If we suppose that the initial white balls are labeled with the integers 1, 2, …, $w$ and let $I_i(m)$ be an indicator random variable with

$$I_i(m) = 1 \quad \text{if ball } i \text{ is in the urn after the } m \text{th R–D cycle}$$

$$= 0 \quad \text{otherwise} \quad (2)$$

we have

$$W(m) = \sum_{i=1}^{w} I_i(m) \quad (3)$$

$$E(W(m)) = w(1 - \rho)^m \quad (4)$$

$$\text{Var} \{W(m)\} = w[(1 - \rho)^m - (1 - \alpha \rho)^m] + w^2[(1 - \alpha \rho)^m - (1 - \rho)^{2m}] \quad (5)$$