A FORMALIZATION OF CONSENSUS INDEX METHODS*

WILLIAM H. E. DAY
Department of Computer Science,
Memorial University of Newfoundland,
St. John's, Newfoundland, Canada A1C 5S7

F. R. McMORRIS
Department of Mathematics,
University of Louisville,
Louisville, KY 40292, U.S.A.

A consensus index method comprises a consensus method and a consensus index that are defined on a common set of objects (e.g. classifications). For each profile of objects, the consensus method returns a consensus object representing information or structure shared among profile objects, while the consensus index returns a quantitative measure of agreement among profile objects. Since the relationship between consensus method and consensus index is poorly understood, we propose simple axioms prescribing it in the most general terms. Many taxonomic consensus index methods violate these axioms because their consensus indices measure consensus object invariants rather than profile agreement. We propose paradigms to obtain consensus index methods that measure agreement and satisfy the axioms. These paradigms salvage concepts underlying consensus index methods violating the axioms.

1. Introduction. In 1972 Adams proposed methods to construct from each profile of trees (representing hierarchical classifications) a consensus tree depicting areas of similarity among the profile trees. He perceived consensus trees to be a complementary alternative to the traditional "numerical index of agreement between trees" (Adams, 1972, p. 390). Subsequently there has been an amazing proliferation of consensus methods (Nelson, 1979; Margush and McMorris, 1981; Sokal and Rohlf, 1981; McMorris and Neumann, 1983; Neumann, 1983; Stinebrickner, 1984) and consensus indices (Mickevich, 1978; Nelson, 1979; Colless, 1980; Nelson and Platnick, 1981; Schuh and Farris, 1981; Rohlf, 1982; Day, 1983; Stinebrickner, 1984): a proliferation stimulated by confusions, disagreements, and uncertainties concerning what consensus methods depict and what consensus indices measure. Thus, for example, consensus indices for trees are understood to measure agreement (Adams, 1972, p. 390; Rohlf, 1982, p. 131), balance (Rohlf et al. 1983, p. 148), information (Mickevich, 1978, p. 147;
Mickevich and Farris, 1981, p. 362; Shao, 1984, pp. 19–21), resolution (Colless, 1980, p. 295), shape (Colless, 1980, p. 295), similarity (Mickevich, 1978, p. 147; Adams, 1972, p. 397), and symmetry (Rohlf, 1982, p. 136). One has the impression that taxonomists do not know (or cannot agree on) what consensus objects should depict or how it should be depicted; they do not know (or cannot agree on) what consensus indices should measure or how it should be quantified. Consequently, taxonomists may not appreciate (or do not articulate) the relationships that might or should exist between consensus method and consensus index.

Our theses are that consensus objects exhibit agreement among profile objects and that consensus indices measure degree of agreement among profile objects. We present a formalization of consensus index methods in which the behavior of consensus method and consensus index is prescribed in the presence of unanimity among profile objects. With respect to this formalization, consensus methods on trees behave consistently with the thesis that they exhibit agreement among profile objects; on the other hand, the behavior of most consensus indices on trees indicates that they do not measure degree of agreement among profile objects. That they do not is not to say that they cannot, and so we describe paradigms to construct consensus indices that measure, according to our formalization, degree of agreement among profile objects.

Our notation and terminology follow that of McMorris and Neumann (1983) and Neumann (1983). \( \mathcal{C} \) denotes a set of objects of interest such as: the set of all partitions of a given set; the set of all unrooted trees with leaves labeled by elements of a given set; or the set of all \( n \)-trees for given positive integer \( n \) (Bobisud and Bobisud, 1972). In taxonomic applications, \( \mathcal{C} \) is usually a set of classifications of interest. \( \mathcal{C}^k \) denotes the \( k \)-fold Cartesian product of \( \mathcal{C} \); its \( k \)-tuples are called profiles and a typical profile is designated \( P = (X_1, \ldots, X_k) \). A consensus method for \( \mathcal{C} \) is modeled as an appropriate function

\[
C: \bigcup_{k=2}^{\infty} \mathcal{C}^k \to \mathcal{C},
\]

while a consensus index for \( \mathcal{C} \) is a function

\[
I: \bigcup_{k=2}^{\infty} \mathcal{C}^k \to [0, 1],
\]

\([0, 1]\) being the closed real interval between 0 and 1. Our consensus methods correspond to the consensus functions of McMorris and Neumann (1983) and to the consensus rules of Neumann (1983). Since taxonomic analyses