MOMENTS AND ORDER STATISTICS OF 
EXTINCTION TIMES IN MULTITYPE BRANCHING 
PROCESSES AND THEIR RELATION TO RANDOM 
SELECTION MODELS

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We investigate the circumstances under which the moments of the order statistics of the extinction times of a set of independent branching processes exist. This extends a result of Schuster and Sigmund, Bull. math. Biol. 46, 11-17, 1984, which was found in a special random selection model. Furthermore we discuss the existence of the expectation of extinction times of multitype branching processes and extend well known results for irreducible processes to the reducible case.

1. Introduction. In a recent paper Schuster and Sigmund (1984a) have investigated the behaviour of the order statistics of extinction times of \( n \) independent critical linear birth and death processes and have related their findings to the neutral theory of evolution, see e.g. Kimura (1982). They found that in their simple random selection model the expectations of all but the last one order statistics were finite. This fact may be interpreted as fixation of one species. In section 3 of this paper we show that their result is true for a much larger class of stochastic models by generalizing their result to arbitrary independent (irreducible) multitype branching processes. We also include a discussion of the subcritical and supercritical cases. In section 2 we extend results of Sewastjanow (1975) concerning moments of extinction times of irreducible multitype branching processes to the reducible case. These results are related to questions originating from recent studies of molecular replication and random selection within the framework of branching processes, see Gassner (1984), Schuster and Sigmund (1984b).

Let us now fix notation, for which Sewastjanow (1975) is the basic reference. Suppose that we are given a population built up of entities from \( n \geq 1 \) different species or types \( I_i, i = 1, \ldots, n \) and let us assume that the evolution of this population is described by a multitype branching process in continuous time. That is if we let \( X_1(t), \ldots, X_n(t) \) denote the size of species \( I_1 \) up to species \( I_n \) at time \( t \geq 0 \) then \( X(t) = [X_1(t), \ldots, X_n(t)] \) is a homogeneous Markov process (more precisely a Markov family) in continuous
time with state space $\mathbb{N}_0^n$ where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ such that its transition function $p(x, y; t)$ satisfies the 'branching condition'

$$p(x, y; t) = [p(e_1, y; t)]^{x_1} \ast [p(e_2, y; t)]^{x_2} \ldots \ast [p(e_n, y; t)]^{x_n}$$

where $e_i$ is the $i$th $(1 \times n)$ unit vector, $x = (x_1, \ldots, x_n)$ and where $\ast$ denotes convolution, see e.g. Athreya and Ney (1972) and Sewastjanow (1975).

We furthermore assume that the transition function satisfies

$$p(e_i, y; t) = \delta_{y_i} + \gamma(e_i, y)t + o(t)$$

for $t \downarrow 0$, where

$$\delta_{y_i} = \prod_{i=1}^{n} \delta_{y_i}^{x_i}$$

and $\delta_{y_i}$ is the Kronecker delta, and that the infinitesimal parameters $\gamma(e_i, y)$ fulfill

$$\sum_{y \in \mathbb{N}_0^n} \gamma(e_i, y) = 0.$$  

As in Sewastjanow (1975) we introduce the generating functions

$$F^x(s; t) = \sum_{y \in \mathbb{N}_0^n} p(x, y; t)s^y, \quad |s| \leq 1,$$

where $s = (s_1, \ldots, s_n), s^y$ is short for

$$\prod_{i=1}^{n} s_{i}^{y_i} \quad \text{and} \quad |s| = \max (|s_1|, \ldots, |s_n|).$$

We set $F^i(s; t) = F^{e_i}(s; t)$ for notational convenience and introduce the vector valued generating function

$$F(s; t) = \{F^1(s; t), \ldots, F^n(s; t)\}.$$  

Of course equation (1) now takes the form

$$F^x(s; t) = \prod_{i=1}^{n} [F^i(s; t)]^{x_i}.$$  

Because of equation (5) we can concentrate on $F^i$ instead of $F^x$ and recover all results derived for starting values $e_i$ for arbitrary starting values $x$. Similarly we define the infinitesimal generating functions

$$f^i(s) = \sum_{y \in \mathbb{N}_0^n} \gamma(e_i, y)s^y, \quad |s| \leq 1,$$