A MATHEMATICAL MODEL OF TRANSMURAL TRANSPORT OF OXYGEN TO THE RETINA

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A mathematical model of transmural transport of oxygen to a metabolizing retina is presented based on the equations of fluid dynamics. The equations of oxygen transfer are derived and then solved subject to the condition that the capillaries begin to transport oxygen at an initial time. The resulting transient analysis gives us insight into how diffusive and filtrative processes lead to the oxygen distributions both inside and outside capillaries. On the other hand, the steady state solution allows us to predict the cutoff intraocular pressure above which no oxygen is transferred to retinal tissue. It also gives quantitative relationships which allow us to postulate how intracapillary hypertension counterbalances elevated intraocular pressures and how low pressure glaucoma may arise from ineffective diffusive and filtrative processes of oxygen transport.

1. Introduction. The problem of transport of matter across capillaries is not new and dates back to the turn of the century (Krogh, 1919). In this early model, steady state diffusive processes were considered between a cylindrical capillary and a concentric mantle of tissue. Even in its simplest description its mathematical solution is complex. In 1943, Block added mass transfer across the capillary wall to the model and in 1953 Sangren and Sheppard simplified it by assuming infinite radial diffusion and zero axial diffusion. In 1950, Opitz and Schneider, in 1957 Kety, and finally in 1960, Thews, solved the Krogh model subject to specified inlet and outlet intravascular concentrations. Their assumptions, however, have been criticized on the grounds that they present an overspecified boundary value problem. In 1959, Aris solved the Krogh model, approximately, by performing a series of integrations over the axial spacial coordinates. He thus obtained partial differential equations for the moments of distribution of the concentration which he then solved. In 1960, Blum carried Block's
approach further by neglecting axial diffusion but including first order tissue metabolism. In that year Bellman et al. proposed a model for drug transfer to tissue which also neglected radial diffusion in addition to neglecting pressure gradients and tissue metabolism. Their model involved the formulation of a complex integrodifferential equation which was not solved for several years (Hass, Kalaba, and Vasudevan, 1974). In 1969, Reneau, Bruley and Knisely and in 1970, Bassingthwaighte, Knopp and Hazelrig applied numerical techniques to the Krogh equations. In 1971 Levitt numerically solved the Krogh equations and compared the results to the Aris solution. All numerical methods, however, have the disadvantage of taking a considerable amount of computer time. More recent analytical solutions are given by Davis, Cooney and Chang in 1974. Several reviews of the Krogh model are available (Bailey, 1967; Goldstick, 1973).

Our objective in the following paper is to formulate and solve an appropriate model of transmural transport of oxygen to the retina of the human eye. To this end, we adopt a one-dimensional Krogh model, but unlike the former models include not only tissue metabolism and time-varying concentrations but also include hydrostatic transmural pressure gradients.

The relationship between elevated intraocular pressure and glaucoma with field loss warrants little comment, but it is interesting to note that some glaucoma patients have low intraocular pressures (so-called “low pressure glaucoma” (Kohner and Hetherington, 1970)) and some hypertensive patients have elevated intraocular pressure without having field loss (Havender, 1970). It is therefore apparent that some balance must exist between transmural pressure and concentration gradients to maintain a healthy retina.

Qualitative discussions concerning the effects of transmural concentration gradients of oxygen and the effects of tissue metabolism in the retina have been given (Wise, Dollery and Henkind, 1971; Dollery, 1968, 1969) but no quantitative model of transport across the retinal capillaries has been proposed. In the following paper we address ourselves to just this problem. Here we restrict ourselves to physiologic oxygen concentrations in blood since elevated concentrations (i.e. breathing 100% O₂) causes severe constriction of vessels (Dollery, 1968, 1969) and low oxygen concentrations causes vasodilation (Dollery, 1968, 1969). We also note that the model loses some meaning at severely elevated intracocular pressures where retinal ischemia is non-linearly enhanced by decreased retinal vascular flow (Dollery, Henkind, Kohner and Patterson, 1968).

2. Mathematical Model. Consider a cylindrical capillary of radius R, length L and wall thickness δ carrying oxygen at concentration $C(Z,t)$ cm⁻³.