A NOTE ON THE SHAPE OF THE ERYTHROCYTE

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Variational calculus is used to derive an equation for the shape of the cross section of a human red blood cell, the objective being the maximization of the surface area/volume ratio. Comparison to previous work is presented.

1. Introduction. It is often stated by authors (Canham, 1970; Fung, 1966) that the biconcave shape of the human red blood cell has been the object of study since its discovery in the seventeenth century. Previous studies provided possible explanations for the biconcave cross-section. Canham (1970) used the equation of Cassini and selected its parameters such that the bending energy of the resulting shape should be a minimum; Szirtes (1971) employed a fourth order polynomial and determined its coefficients so that the surface area to volume ratio should be maximized in order to provide for the most efficient transport of oxygen. He compared the results to the actual reported shape of the red blood cell (Ponder, 1948) and found good agreement.

In the present note, following Szirtes, it is assumed that the main purpose for the existence of the red blood cell is oxygen transport; mathematical expressions for the surface area and volume are derived and variational calculus is employed to obtain the cross-sectional shape of the blood cell by maximizing the ratio of surface area to volume. The purpose of this approach is to emphasize the reliance on an analytical technique rather than the numerical technique of curve fitting, employed by previous authors (Canham, 1970; Szirtes, 1971). It is shown that the shape obtained here compares well to the actual shape.
2. Analysis. The volume of a body of rotational symmetry about the $y$ axis—see Figure 1—is given by

$$V = 4\pi \int_0^b y \, dx,$$

and its surface area by

$$S = 4\pi \int_0^b x \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} \, dx.$$ (2)

For the following computations it is believed to be reasonable to assume that the shape of the cross section near the edge may be approximated by a circle (Szirtes, 1971), therefore

$$y = \sqrt{r^2 - (x - a)^2} \quad \text{for} \quad x_1 \leq x \leq b,$$ (3)

where $r = 1.2 \mu$ (Ponder, 1948).

Taking the volume to be a constant and using it as a constraint, the Euler-Lagrange equation to maximize the surface area for the range $0 \leq x \leq x_1$ is obtained as

$$\frac{d^2y}{dx^2} + \frac{1}{x} \left[ \left( \frac{dy}{dx} \right)^3 + \frac{dy}{dx} \right] - \lambda \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2} = 0,$$ (4)

where $\lambda$ is a Lagrange multiplier.

Equation (4) is a second order, ordinary non-linear differential equation whose solution is subject to the boundary conditions