IONIC CONCENTRATION PROFILES
AND CHARGE DISTRIBUTION FOR
THE DOMAIN OF A
POLARIZED SPHERICAL CELL

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Solutions of the Poisson–Boltzmann equation yield potential profiles and equilibrium distributions of ions on either side of a spherical shell membrane across which there exists a separation of ionic charges. For the special case in which the membrane is permeable to only one ion the total charge separation is analyzed in terms of the potential difference given by the Nernst equation. Potential profiles and ionic charge distributions are also given for situations involving a uniform distribution of fixed charges within the membrane.

Recent calculations of Rehm et al. (1976) have indicated that under equilibrium conditions a typical ionic pump can separate electrical charges of opposite sign across the membrane of a small vesicle to an extent such that it should be possible to detect by chemical methods the resultant change in either internal or external concentration of a given ionic species. It is of interest then to determine the profiles of ionic concentrations as a function of position both intracellularly and extracellularly at regions where net charge density is finite.

Original one dimensional analyses stem from the classical formulations of Gouy (1910) and Chapman (1913) who applied Poisson–Boltzmann statistical considerations to a system in which ionic solutions are subject to the forces exerted by a planar surface of fixed charges. The resultant equilibrium distribution of ions has been the subject of an investigation by Chandler, Hodgkin and Meves (1965) who considered the consequences of assuming that
squid giant fiber axoplasm is subjected to forces exerted by a planar surface
distribution of charges fixed on the inner surface of the membrane.

Extension of this type of analysis to three dimensional surfaces has involved
ionic environments external to both the charged cylinder (Abramson et al.,
1942) and the charged sphere (Loeb et al., 1961).

In this particular presentation attention will be given potential profiles and
equilibrium distributions of ions on both sides of a spherical shell membrane
across which there exists a separation of electrical charges. Radial dimensions
are shown in the inset diagram of Figure 1. Data pertaining to the ionic
environment of the membrane are indicated in the Appendix. It is assumed that
a pump introduces a catonic charge \( q \) on the outer surface of the sphere leaving
an anionic charge inside the vesicle under equilibrium conditions such that no
leaks exist across the membrane. Later provisions will take into account steady
states involving pumps and leaks. Thickness of the membrane is \( (r_0 - b) \) so that
the negative charge \( q \) is enclosed within the sphere of radius \( b \).

An inwardly directed electric field will then exist in the external solution at
regions where catonic charge density is finite. Since radial symmetry must hold
throughout the system it is obvious that the electric field induced by the
enclosed negative charge must constrain the outer catonic charge to lie in a
concentric shell just outside the radius \( r_0 \). Within this outer shell of positive
ions each ion will be subjected to a balance of forces involving electric field and
concentration gradient for the ion in question.

As a condition for equilibrium the current density for any given ion is zero at
any point so that in terms of the Nernst–Planck equation for a monovalent
cation such as sodium the current density would be

\[
I_{Na} = - (Na) \mu_{Na} \left[ RT \frac{\ln(Na)}{dr} + F \frac{dV}{dr} \right] = 0,
\]

where \( (Na) \) is sodium concentration at any radius point \( r \), \( V \) is the potential and
\( \mu_{Na} \) is the electrical mobility in terms of velocity per unit electric field. On
multiplying by \( dr \) and integrating between any two points \( r \) and \( a \) we obtain

\[
\left[ \frac{RT}{F} \right] \int_r^a \frac{d\ln(Na)}{dV} = - \int_r^a dV,
\]

from which

\[
[Na] = [Na]_a e^{-F(V - V_a)/RT},
\]

where \([Na]_a\) at any point \( r \) is evaluated in terms of \( V \) at that same point. \( V_a \) and
\([Na]_a\) denote potential and concentration at a remote reference point \( a \) in the