NOTE

APPLICATION OF HAMILTON-JACOBI THEORY TO THE
LOTKA-VOLTERRA OSCILLATOR

RANABIR DUTT
Department of Physics, Visva-Bharati University,
Santiniketan 731235,
West Bengal, India

It is known that the Lotka-Volterra coupled nonlinear differential equations for a two-
species prey-predator ecosystem possess a periodic solution, although its exact form is not
yet obtained analytically. The conventional linearization approximation for solving these
nonlinear equations leads to a harmonic oscillator whose frequency depends only on the
intraspecific coefficients. We propose here a prescription for obtaining nonlinear correction
to the linear frequency by using the Hamilton-Jacobi canonical formalism of classical
mechanics. It is found that the first-order correction, which also involves interspecific
parameters, exhibits the basic qualitative features of the nonlinearity.

1. Introduction. There have been several recent papers (Montroll, 1972;
Grasman and Veling, 1973; Frame, 1974; Trubatch and Franco, 1974; Dutt,
Ghosh and Karmakar, 1975) concerned with the determination of parameters
associated with the nonlinear Lotka-Volterra (LV) equations for conflicting
populations. All these papers produce estimates of the period of oscillation
of the LV oscillator. All use different techniques and some certain restrictive
assumptions to compute corrections to the period of the linearized system.
The most elegant method was given by Frame (1974) who derived an exact
period for the two-species LV system in terms of modified Bessel functions.

In this paper, we propose an alternative method to obtain an approximate
period of oscillation in nonlinear conservative systems. We use the Hamilton–
Jacobi canonical formalism, originally devised to handle the problem of classical
mechanics (Goldstein, 1962). In Section 2, we shall show how the LV equations
can be formally put into appropriate canonical form. We then present the method of introducing action and angle variables by means of successive canonical transformations for successive orders in the expansion of the Hamiltonian and obtain nonlinear corrections, up to certain orders, to the linear frequency. Our approach is analogous to the method used by Lewis and others (Lewis, 1967, 1968; Symon, 1970). In Section 3, we present a brief discussion about our results, mentioning other possible application of the method.

2. The Formalism. In dealing with the problems involving periodicity, the Hamilton–Jacobi canonical theory has a distinct advantage over the conventional methods of classical mechanics. In this approach, one introduces action and angle variables through canonical transformations in such a way that the angle variable becomes cyclic. One then obtains the frequency of oscillation by taking the derivative of the Hamiltonian with respect to the action variable. One may thus bypass the difficulty in obtaining the complete solutions of the equations of motion, if these are not required.

To see how we can employ this trick to the biological oscillators, we consider the two-species LV system. The LV equations used to model the prey–predator problem in the two-species case are given by

\[
\frac{dN_1}{dt} = \alpha_1 N_1 - \lambda_1 N_1 N_2, \quad \frac{dN_2}{dt} = -\alpha_2 N_2 + \lambda_2 N_1 N_2,
\]

where \(N_i (i = 1, 2)\) is the number of individuals of species \(i\) at a given time, \(\alpha_i\) is the innate capacity for increase per individual (intraspecific coefficients) and \(\lambda_i\) is the interspecific coefficient (niche overlap parameter). The nontrivial equilibrium values \(\{q_i\}\) of \(N_i\), such that \(dN_1/dt = dN_2/dt = 0\), are

\[q_1 = \frac{\alpha_2}{\lambda_2}, \quad q_2 = \frac{\alpha_1}{\lambda_1}.
\]

Choosing a new set of variables

\[
p(t) = \log_e (N_1(t)/q_1), \quad x(t) = \log_e (N_2(t)/q_2),
\]

the equations (1) can be reduced to the form

\[
\dot{p} = \alpha_1 (1-e^x), \quad \dot{x} = -\alpha_2 (1-e^p),
\]

a form which is easily seen to be canonical with the Hamiltonian

\[
H(x, p) = \alpha_1 (e^x - 1 - x) + \alpha_2 (e^p - 1 - p).
\]

From (4) and (5), one can check that the Hamiltonian is a constant of motion.

The variables \(x\) and \(p\) are the amplitudes of oscillation of the predator and prey species about their respective equilibrium population levels, \(q_2\) and \(q_1\).