ANALYSIS OF THE BEHAVIOUR OF KAUFFMAN
BINARY NETWORKS—II. THE STATE
CYCLE FRACTION FOR NETWORKS
OF DIFFERENT CONNECTIVITIES

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The state-transition matrix description of Kauffman binary networks described in the
previous paper is further developed to obtain an analytical expression for the fraction of
states involved in limit cycles as a function of the network size and connectivity. The result
obtained for totally connected networks agrees with that derived from quite different
considerations by other workers. For low connectivity networks the results are in
qualitative agreement with the experimental data of Kauffman but there is a quantitative
discrepancy which remains to be resolved.

1. Introduction. In the previous paper (Sherlock, 1979; subsequently
referred to as Paper I) it was shown that the dynamical behaviour of a
Kauffman network of binary elements (Kauffman, 1969) could be repre-
sented by a state transition matrix \( A \) having the property

\[ \mathbf{q}(t + \tau) = A \mathbf{q}(t), \]

where \( \mathbf{q} \) is an appropriately defined state vector. Various properties of
the matrix \( A \) were demonstrated, and in particular examples were given to
indicate how quite general probabilistic considerations of the distribution
of 0- and 1-valued elements in \( A \) could lead to quantitative predictions
about the network dynamics. In this paper these ideas are further
developed in an attempt to understand the central and most surprising
feature of Kauffman's findings, namely the smallness of the set \( Q_c \) of states
involved in limit cycles for networks of low connectivity. As shown in
Paper I, Kauffman's (1969) results for a net of connectivity \( \kappa = 2 \) imply \( N_c = |Q_c| \approx \mu \) where \( \mu \) is the number of elements in the net. Since the total number of states possible is \( N = 2^\mu \) then the fraction of cycle states \( F_c \approx \mu / 2^\mu \) which is an extremely small number for any "biologically reasonable" value of \( \mu \) (i.e. \( \mu > 1000 \)).

The principle of the calculation to be presented here has been outlined in Paper I, Section 3, namely the identification of states with no predecessors by the occurrence of rows of \( A \) containing no 1-valued elements, and their subsequent removal on the grounds that they cannot be involved in any limit cycle.

2. The Probability of Two States having the Same Successor. Recall from Paper I that the \( N \) columns of the state transition matrix \( A \) are the state vectors \( q \) of those states which have a predecessor. The vectors \( q \) have dimension \( N \) (so that \( A \) is an \( N \times N \) matrix) and a single 1-valued element (the others being 0), the position \( k \) of which defines the state \( q_k \) (\( 0 < k < N - 1 \)). Hence if a given state \( q_k \) has two (or more) predecessors then the column \( q_k \) will occur two (or more) times and this inevitably results in other states having no predecessors (since \( A \) can now not contain the complete set of \( N \) different state vectors as its columns). Thus calculation of the probable number of states with zero predecessors in a net whose detailed specification ("wiring diagram" and assignment of Boolean functions to net elements) is unknown is dependent on a knowledge of the probability \( P_s \) of two arbitrarily chosen states having the same successor. This is now calculated.

Consider an arbitrarily chosen element in a Kauffman net of size \( \mu \) and connectivity \( \kappa \), and let the net be in some state \( q' \), say. Since each element in the net has \( \kappa \) inputs the chosen element will sample \( \kappa \) of the element outputs. Now consider the net in some different state \( q'' \). In general some of the \( \mu \) element outputs will have changed and others will be the same. Since the outputs are two-valued and \( q' \) and \( q'' \) are selected from the complete set of states (the complete set of \( 2^\mu \) permutations of element output values) then the probability that any chosen output is unchanged in going from \( q' \) to \( q'' \) is \( \frac{1}{2} \). Thus the probability that all the inputs of the chosen element are unchanged in going from \( q' \) to \( q'' \) is \( P_e = \left( \frac{1}{2} \right)^{\kappa} \). Clearly, if the element inputs are unchanged then so is its output, but depending on the Boolean function implemented by the element its output may also remain unchanged even though its inputs do change, i.e.

\[
P_e = P_\kappa + (1 - P_\kappa) \cdot P_B : P_\kappa = 2^{-\kappa}
\]

where \( P_e \) is the total probability of the element output being unchanged in going from one arbitrary state to another and \( P_B \) is the probability that the