ON STAR PRODUCT FRACTAL SURFACES 
AND THEIR DIMENSIONS *

Xie Heping (谢和平)\textsuperscript{1}, Feng Zhigang (冯志刚)\textsuperscript{1,2}, Chen Zhida (陈至达)\textsuperscript{3}

(1. Institute of Rock Mechanics & Fractals, China University of Mining and Technology, Beijing 100083, P R China; 
2. Department of Mathematics, Zhenjiang Teacher’s College, Zhenjiang 212003, P R China; 
3. Department of Fundamental Science, China University of Mining and Technology, Beijing 100083, P R China)

Abstract: In this paper, by using fractal curves, a family of fractal surfaces are defined. Each fractal surface of this family is called Star Product Fractal Surface (SPFS). A theorem of the dimensions of the SPFS is strictly proved. The relationship between the dimensions of the SPFS and the dimensions of the fractal curves constructing the SPFS is obtained.

Key words: dimension; fractal curve; fractal surface, SPFS

CLC number: 018 Document code: A

Introduction

Generally speaking, fracture surfaces of materials are rough, irregular and random. The fracture and weakness in rock, concrete, ceramics and metal significantly affect the deformation, strength, and conductivity of these materials. Many paper (such as [1-3]) focused on the fracture surfaces of materials. They revealed that the fracture surfaces have fractal features and the fractal surfaces can be used to simulate the fracture surface. The fractal dimensions can be used to characterize the roughness of the fracture surfaces, and then there are some relationships between the dimensions of the fracture surfaces and macro-mechanical quantities that cause the materials to be damaged and collapsed. Therefore, fractal surfaces, as a special kind of fractal sets, are widely used in nature research (cf. [4,6]).

Fractal surface is a special fractal set in three-dimensional Euclidean space \( R^3 \). Generally a fracture surface of a material can be expressed by a bivariate function \( z = f(x, y) \) in a field \( E \) of a coordinate plane \( xOy \). If the surface has some fractal features, as a general rule, the function is continuous everywhere but differentiable nowhere in the field \( E \). A family of fractal surfaces are simple that can be expressed by a cartesian product of a fractal curve \( A = \{(x, z): x \in [c, d], z = g(x)\} \) in the plane \( R^2 \) and a interval \( B = [a, b] \) in the \( R \), i.e.

\[ F = A \times B = \{(x, y, z): x \in [c, d], y \in [a, b], z = g(x)\}. \]

Received date: 1999-07-09

Foundation item: the Outstanding Youth Science Foundation of China (59425003)
It means that the fractal surface is produced by moving the curve $A$ along a straight line. Another family of fractal surfaces are called Brownian surfaces (cf. [4]). For $0 < \alpha < 1$, a Brownian function $f: R^2 \rightarrow R$ with exponent $\alpha$ is a random function, which satisfies: 1) with probability 1, $f(0,0) = 0$ and $f(x,y)$ is continuous for any $(x,y)$; 2) for every $(x,y)$, $(h,k) \in R^2$, the difference $f(x+h, y+k) - f(x,y)$ yield a normal distribution with the mean value 0 and the mean square deviation of $(h^2 + k^2)^{\alpha}$, namely

$$P(f(x+h, y+k) - f(x,y) \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \frac{1}{(h^2 + k^2)^{\alpha/2}} e^{\frac{-r^2}{2(h^2 + k^2)^{\alpha}}} dr.$$  

The set $\{(x, y, z): (x, y) \in R^2, z = f(x,y)\}$ is called Brownian surface with exponent $\alpha$.

Compared with artificial mathematical fractals, the intrinsic physical fractals in the nature possess randomness, complexity and scale limits (It means that there are fractal features only in a positive interval $[r, \xi]$). There is no pure mathematical fractal in the nature. During the proceeding of the science research, for an object with some fractal features, some approximational fractal models can be firstly proposed, and then to be used to analyse and study the object. It is not surprised that the models of an object may be different due to the variation of researchers’ view points. For fracture surfaces, at first, all researchers recognized the fracture surface as a self-similar fractal. With the study becoming deeper and deeper, researchers discovered that there is some contradiction in describing the fracture surfaces with self-similar fractals. Then self-affine fractal approach is proposed to simulate the fracture surfaces. In fact, the natural fracture surfaces can neither be strictly self-affine fractals nor be strictly self-similar fractals. It remains to make a further study to answer the question whether there are better approaches to the fracture surfaces.

Because of the irregularity and randomness of the fracture surface, it is very difficult to calculate its dimension. So the approximate computation methods are used. As is well-known, every sectional profile of the fractal surface is a fractal curve, the fractal dimension of the sectional profile is used to reflect the roughness of the fracture surface to some extent, but the dimension of the sectional profile lies between 1 and 2, while the dimension of the fracture surface must lie between 2 and 3. Therefore, the dimension of the sectional profile plus one can approximately serve as the dimension of the fracture surface. Obviously this approach is very superficial. There are some other methods similar to this one. For example, the dimension of the contour of the fracture surface plus one serves as the dimension of the fracture surface. Because the methods are different, the calculated dimensions are different too. Sometimes these differences are hard to believe. Therefore, when using fractal dimension to interpret some natural phenomena, sometimes unreasonable or even self-contradictory results might be obtained.

Based on the above-mentioned reason, many researchers are involved in studying and improving the computational method of the dimension of fractal surface. For any sets $A \subseteq R^n$, $B \subseteq R^m$, cartesian product $A \times B \subseteq R^{n+m}$, a relation of dimensions can be given by Falconer[4]

$$\dim (A \times B) \leq \dim A + \dim B.$$  

If a fractal surface $F$ can be expressed as a cartesian product of a fractal curve in $R^2$ and an interval in $R^1$, it means that the fractal surface $F$ can be obtained by moving fractal curve $A$ along a straight line, then $\dim F \leq \dim A + 1$. For a Brownian surface with exponent $\alpha$, its dimension is $3 - \alpha$ with probability 1, namely $P(\dim F = 3 - \alpha) = 1$ (cf. [4]). For a fractal interpolation