VARIATIONAL PRINCIPLES OF FLUID FULL-FILLED ELASTIC SOLIDS

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Abstract: The generalized variational principles of isothermal quasi-static fluid full-filled elastic solids are established by using Variational Integral Method. Then by introducing constraints, several kinds of variational principles are worked out, including five-field variable, four-field variable, three-field variable and two-field variable formulations. Some new variational principles are presented besides the principles noted in the previous works. Based on variational principles, finite element models can be set up.

Key words: fluid full-filled elastic solids; variational integral method; variational principles; generalized variational principles

Introduction

Many practical problems in engineering involve analysis of fluid full-filled elastic solids. Energy exploration and utilization are two examples. The field equations of Biot’s statical and dynamical theory of fluid full-filled elastic solids were established in Refs. [1, 2]. Because it is difficult to get exact answers, numerical methods are adopted, especially the finite element method.

The element method based on variational principles is applied extensively. Ghaboussi and Wilson derived variational principles on the basis of Biot’s equations and set up a finite element method[3]. Sandhu and Pister presented some variational principles for dynamic analysis of the same media[4, 5]. This paper, starting from Biot’s equations, presents a number of variational principles for fluid full-filled elastic solids including principles on five-field variables, four-field variables, three-field variables and two-field variables. Some new principles are presented besides all the known results in previous works.

1 Basic Equations and Conditions

To the porous media which occupies a volume V with a surface S, the governing equations are as follows:

\[
\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}),
\]

\[
\sigma_{ij,i} + F_i = 0,
\]

\[
\varepsilon_{ij} = \frac{1}{2\mu} \left( \sigma_{ij} - \frac{\gamma}{1 + \gamma} \sigma_{kk} \delta_{ij} \right) + \frac{1}{3H} p \delta_{ij},
\]

\[
\theta = \frac{1}{3H} \sigma_{kk} + \frac{1}{R} p,
\]

or

\[
\sigma_{ij} = -\lambda \delta_{ij} + H_{ijkl} \varepsilon_{kl},
\]

\[
\theta = \lambda \varepsilon_{kk} + \frac{1}{Q} p,
\]

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\[ v_i = -k(p_{i,i} + f_i), \]
\[ v_{i,i} = -\theta, \]

boundary conditions:
\[ u_i = u_i^*, \]
\[ \sigma_{ij,i} = \sigma_{ij}^*, \]
\[ v_{p,i} = v_{p}^*, \]
\[ p = p^*, \]

initial conditions:
\[ t = 0^*, \theta(x_1, x_2, x_3, 0^*) = 0, \]

where \( u_i \) is the displacement vector, \( \sigma_{ij} \) is the stress tensor, \( \varepsilon_{ij} \) is the strain tensor, \( p \) is the change in pore pressure, \( \theta \) is the increase in fluid content as the result of change in pore size, \( v_i \) is the velocity of liquid in the porous media.

\( u_i^*, p_i^*, v_i^* \) and \( p^* \) are prescribed values on \( S_u, S_p, S_v \) and \( S_p; F_i \) is the total body force per unit volume, \( f_i \) is the body force per unit volume due to the fluid; \( \mu \) is the shear modulus, \( \gamma \) is Poisson's ratio under isothermal condition; \( K \) is the coefficient of permeability; \( H \) and \( R \) are physical constants of the porous media.

Where
\[ \lambda = \frac{2\mu(1 + \gamma)}{3H(1 - 2\gamma)}, \quad \frac{1}{Q} = \frac{1}{R} - \frac{\lambda}{H}, \quad H_{ijkl} = 2\mu \left( \partial_q \delta_{kl} + \frac{\gamma}{1 + \gamma} \delta_{qq} \delta_{kl} \right). \]

Surface \( S \) requires
\[ S_u + S_v = S_p + S_v = S. \]

By using Laplace transformation, the governing equations may be written as:

\[ \varepsilon_{ij} = \frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i}) \quad \text{(in V)}, \]  
\[ \bar{u}_i = \bar{u}_i^* \quad \text{(on } S_u), \]  
\[ \bar{\sigma}_{ij} = \bar{F}_i = 0 \quad \text{(in V)}, \]  
\[ \bar{\sigma}_{ij} = \bar{F}_i^* \quad \text{(on } S_p), \]  
\[ \bar{\sigma}_{ij} = -\lambda \bar{p} \bar{\sigma}_{ij} + H_{ijkl} \varepsilon_{kl} \quad \text{(in V)}, \]  
\[ \frac{1}{H} \bar{v}_{i,i} + \lambda \bar{e}_{kk} + \frac{1}{Q} \bar{p} = 0 \quad \text{(in V)}, \]

or

\[ \bar{\varepsilon}_{ij} = -\beta \bar{p} \bar{\varepsilon}_{ij} + C_{ijkl} \bar{\sigma}_{kl} \quad \text{(in V)}, \]  
\[ \frac{1}{H} \bar{v}_{i,i} + \beta \bar{e}_{kk} + \frac{1}{R} \bar{p} = 0 \quad \text{(in V)}, \]  
\[ \bar{v}_i = -k(\bar{g}_i + \bar{f}_i) \quad \text{(in V)}, \]  
\[ \bar{g}_i = \bar{p}_{i,i} \quad \text{(in V)}, \]  
\[ \bar{p} = \bar{p}^* \quad \text{(on } S_p), \]  
\[ \bar{v}_{p,i} = \bar{v}_p^* \quad \text{(on } S_v), \]

where \( \bar{\varepsilon}_{ij} = \varepsilon_{ij}, \) \( l \) is parameter in Laplace transformation.

2 Generalized Variational Principles with Six Kinds of Independent Variables

Taking variational integral on Eqs. (1) ~ (10), we obtain:

\[ \int_0^1 q_i \delta \Pi_6 = \int_0^s \int_V (\bar{u}_{i,j} - \varepsilon_{ij}) \delta \bar{\sigma}_{q} dV - \int_0^s \int_V (\bar{\sigma}_{q,i} - \bar{F}_i) \delta \bar{u}_i dV - \]

\[ \int_0^s \int_V \bar{v}_i \delta \bar{g}_i dV - \int_0^s \int_V \bar{g}_i \delta \bar{v}_i dV - \int_0^s \int_V \bar{p} \delta \bar{p} dV - \int_0^s \int_V \bar{p}_{i,i} \delta \bar{p}_{i} dV - \]

\[ \int_0^s \int_V \bar{v}_p \delta \bar{v}_p dV - \int_0^s \int_V \bar{v}_{p,i} \delta \bar{v}_{p,i} dV - \]

\[ \int_0^s \int_V \bar{p} \delta \bar{p} dV - \int_0^s \int_V \bar{p}_{i,i} \delta \bar{p}_{i} dV - \int_0^s \int_V \bar{v}_p \delta \bar{v}_p dV - \int_0^s \int_V \bar{v}_{p,i} \delta \bar{v}_{p,i} dV - \]

\[ \int_0^s \int_V \bar{p} \delta \bar{p} dV - \int_0^s \int_V \bar{p}_{i,i} \delta \bar{p}_{i} dV - \int_0^s \int_V \bar{v}_p \delta \bar{v}_p dV - \int_0^s \int_V \bar{v}_{p,i} \delta \bar{v}_{p,i} dV - \]

\[ \int_0^s \int_V \bar{p} \delta \bar{p} dV - \int_0^s \int_V \bar{p}_{i,i} \delta \bar{p}_{i} dV - \int_0^s \int_V \bar{v}_p \delta \bar{v}_p dV - \int_0^s \int_V \bar{v}_{p,i} \delta \bar{v}_{p,i} dV - \]

\[ \int_0^s \int_V \bar{p} \delta \bar{p} dV - \int_0^s \int_V \bar{p}_{i,i} \delta \bar{p}_{i} dV - \int_0^s \int_V \bar{v}_p \delta \bar{v}_p dV - \int_0^s \int_V \bar{v}_{p,i} \delta \bar{v}_{p,i} dV - \]

\[ \int_0^s \int_V \bar{p} \delta \bar{p} dV - \int_0^s \int_V \bar{p}_{i,i} \delta \bar{p}_{i} dV - \int_0^s \int_V \bar{v}_p \delta \bar{v}_p dV - \int_0^s \int_V \bar{v}_{p,i} \delta \bar{v}_{p,i} dV - \]