

**PATH-INDEPENDENT J-INTEGRAL AND ITS DUAL FORM IN ELASTIC-PLASTIC SOLIDS WITH FINITE DISPLACEMENTS**

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Abstract: In this paper, based on energy variational principles of elastic-plastic solids, the path-independent J-integral and its dual form in elastic-plastic solids with finite displacements are presented. Whose testification is given there after.

Key words: finite displacements; elastic-plastic solids; J-integral

Introduction

J-integral was first proposed by Rice in 1968 and Rice pointed out that the physical meaning of J-integral is equivalent to the energy release rate defined as the negative of change in the potential energy with respect to unit advance of the crack in its plane (per unit thickness). Knowles and Sterberg, basing on two dimension (2D) J-integral, defined seven curve square independent integrals in 1972. Ouyang Chang, Lu Meizi and Chen Zhida et al. have developed and extended those work based on dynamics and three dimension (3D). The proposal of J-integral had saved researchers from the trouble in directly finding out the analytic solutions on the crack tip and suggested them clarifying the relationship between J-integral and fracture. For example, there is a kind of linear relationship between crack open displacement (COD) \( \partial \) and J-integral in the condition of 2D elastic-plastic solids. And in the condition of 3D, numerical calculation, given by Brocks and Nioack in 1988, had presented that this linear relationship is still existent, but in theory there needs further reasoning. The research of the relationship between J-integral and fracture is one of the key work in fracture and the foundation of J-integral has set up the reliable mathematical base for numerical calculation of fracture. At present, the research of J-integral is still one of the hot-points of surface fracture, singularity of crack tip and computing dynamic fracture etc.

J-integral, proposed by Rice, is a kind of potential form integral based on small scale displacement (SSD). In elastic mechanics, we recognize that there is a sort of dual relationship between potential energy and residual energy, so it is possible to find a new kind of path-independent J-integral based on residual energy, named dual form of J-integral. On the other hand, in general there are large strains and displacements on the crack tip of elastic-plastic solids, so it is also necessary to care about the path-independent integral on the crack tip. We think that our present work will be useful to expose further the conservation laws on the crack tip of elastic-plastic solids with finite displacements.

1 J-Integral and Its Dual Form in Elastic-Plastic Solids with Finite Displacements

There is a kind of dual relationship between potential and residual energy in solids according
to the variational principles of elastic-plastic mechanics. In general, the \( J \)-integral and its dual form \( J' \), in the condition of elastic-plastic solids with finite displacements and with body forces neglected, can be presented:

\[
J = \int_{\Gamma} A d\gamma - \int_{\Gamma} T_i \cdot \frac{\partial u_i}{\partial x} ds, \quad (1)
\]

\[
J' = \int_{\Gamma} \left( B + \frac{1}{2} \cdot u_{k,i} \cdot u_{k,i} \cdot \sigma_{ij} \right) d\gamma - \int_{\Gamma} \frac{\partial T_i}{\partial x} \cdot u_i ds, \quad (2)
\]

where \( \Gamma \) is any contour surrounding the crack tip; \( T_i \) and \( u_i \) are the traction and displacement on \( \Gamma \), respectively; \( \frac{\partial u_i}{\partial x} \) and \( \frac{\partial T_i}{\partial x} \) are the spatial gradient of the displacement and traction in the direction of \( x \)-axis, respectively; \( ds \) is the arc length along \( \Gamma \) as is shown in Fig. 1; \( A \) and \( B \) are the strain energy density and residual density, respectively. And

\[
A = \int_0^{\varepsilon_y} \sigma_y d\varepsilon_y, \quad (3)
\]

\[
B = \int_0^{\varepsilon_y} \varepsilon_y d\sigma_y, \quad (4)
\]

where every variable in formulas (1) – (4) is suited to the governing equations of elastic-plastic solids with finite displacements.

\( J \)-integral and its dual form, based on the principles of variation, are tenable in both cases of SSD and finite displacements. And \( J \)-integral shares in the same form in either of two cases and its dual form \( J' \) can return to the form of SSD by erasing the higher order term.

2 Path-Independence \( J \)-Integral and Its Dual Form in Solids with Finite Displacements

Considering the elastic-plastic solids (without unloading) with finite displacements and using the coordinate system of Lagrange, we can determine the position of points in solids after being deformed by the previous coordinate of points and the governing equations of solids can be drawn as follows:

1) equilibrium

\[
\left[ (\delta_{ik} + u_{i,k}) \sigma_{ij} \right]_{,j} + f_i = 0; \quad (5)
\]

2) relationship between strain and displacement

\[
\varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} + u_{k,i} \cdot u_{k,j} \right); \quad (6)
\]

3) relationship between stress and strain (without unloading)

\[
\sigma_{ij} = \frac{\partial A(\varepsilon_{ij})}{\partial \varepsilon_{ij}}, \quad \varepsilon_{ij} = \frac{\partial B(\sigma_{ij})}{\partial \sigma_{ij}}; \quad (7)
\]

4) force boundary condition

\[
\left( \delta_{ik} + u_{i,k} \right) \cdot \sigma_{ij} \cdot n_j = p_i; \quad (8)
\]