ON THE PROBABILITY OF REACHING A THRESHOLD IN A STOCHASTIC MAMMILLARY SYSTEM

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The multivariate distribution over time of a particular stochastic mammillary compartmental model is obtained for any point in time. The maximum expectation of the peripheral compartments is then derived and used to determine lower bounds on the probability that the maximum of the peripheral compartments reaches any arbitrary threshold level. A bound on the probability is illustrated by an example and some of its implications are explored.

1. Introduction. The purpose of this paper is to explore certain properties of a stochastic mammillary system with particular reference to the probability of reaching a threshold value in any of the identical peripheral compartments. The mammillary compartmental system is a well-known model which has been widely used in biomedical modelling (see e.g. Sheppard, 1962) and similar kinetics have also been employed to describe other physical, ecological and social systems. However, the incorporation of certain stochastic variation into general multicompartent models has been a recent development with the papers of Matis and Hartley (1971) and Thakur et al. (1973). The present paper examines a consequence of such variation for a particular mammillary system.

We consider first the special mammillary system with $k$ peripheral compartments as illustrated in Figure 1 and defer until later the generalization to other mammillary models. Let $X_i(t)$ for $i = 1, \ldots, k$ denote the number of particles in each of the peripheral compartments at time $t$ with $X_{k+1}(t)$ representing the...
central compartment and \( X_{k+2}(t) \) the number which have left the system. Let
the system be pulse labelled with \( N \) particles in the central compartment, i.e. \( X_{k+1}(0) = N \) and \( X_i(0) = 0 \) for \( i = 1, \ldots, k \) and for \( i = k + 2 \). The

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\begin{align*}
&\text{Figure 1. A simple mammillary compartmental model with } k \text{ peripheral compartments} \\
&\text{transitions to the compartments are assumed to be irreversible and the transition coefficients are identical. These assumptions give rise to the following transition probabilities for small } \Delta t: \\
&\text{Prob. \{a given particle in comp. } k + 1 \text{ leaves the system in the interval } (t, t + \Delta t)\} = \beta \Delta t; \\
&\text{Prob. \{a given particle in comp. } k + 1 \text{ transfers to comp. } i \text{ in the interval } (t, t + \Delta t)\} = \alpha \Delta t, \text{ for } i = 1, \ldots, k; \text{ and} \\
&\text{Prob. \{a given particle in comp. } i \text{ leaves the system in the interval } (t, t + \Delta t)\} \\
&\quad = \alpha \Delta t, \text{ for } i = 1, \ldots, k. \\
\end{align*}
\]

This special model generalizes the single-hit radiation model which was originally proposed by Lea (1946) and which has recently been used as a non-parametric bioassay model (Mantel and Bryan, 1961). In this application, the central compartment is the blood stream and one is interested in describing the number of particles which enter into any of \( k \) groups of cells, where \( k \) is typically very large. This mammillary model may also be used in the multiple population mark-recapture problem in wildlife science in which animals are marked and recaptured in several adjacent areas (see e.g. Arnason, 1972). In a current study, a number of insects (boll weevils) are released in a central zone and one desires information on the numbers which migrate to the \( k \) adjacent fields, where \( k \) is a small number. Whether \( k \) is large or small, the primary interest in these problems centers on the probability that the number in any peripheral compartment reaches a specific threshold (or infestation) number, say \( s \).

2. Distribution of Particles at Fixed \( t \). The first step in investigating the maximum in the peripheral compartments is to determine the multivariate dis-