ASYMPTOTIC METHODS IN DYNAMIC CONTACT PROBLEMS FOR AN ELASTIC HALF-SPACE

E. V. Kovalenko 1 and V. B. Zelentsov 2

UDC 539.3

The following dynamic contact problems of the theory of elasticity are considered: (1) the problem of antiplane shear of an elastic half-space by a punch and (2) the plane problem of pressing of a punch into an elastic half-plane. We assume that at time \( t = 0 \), a force that varies arbitrarily in time is applied to the punch.

To solve these dynamic problems, we apply the Laplace–Carson transform with respect to time and the Fourier transform with respect to the spatial coordinate. As a result, the Laplace–Carson-transformed contact stress problems are reduced to Fredholm integral equations of the first kind of the convolution type on a finite interval, with kernels depending on the dimensionless parameter \( \lambda \in (0, \infty) \) related to time.

To solve these equations, we used the methods of [1, 2]. For large and small values of \( \lambda \), which correspond to large and small times of interaction of a punch and a half-space, simple analytical solutions are obtained in several forms, each of which is effective in its region of variation of the parameter \( \lambda \). Calculations have shown that these regions overlap the entire possible range of variation of \( \lambda \). To obtain the final solution of the problems, in the resulting formulas, we go over from the Laplace–Carson transform of unknown functions to their originals.

1. Let an isotropic elastic half-space be subjected to pure shear under the effect of an infinite undeformable band of width \( 2a \) loaded along its generatrix by a shearing force \( T(t) = T_0 f(t) \) \([f(t) \text{ is a bounded function with a finite number of discontinuity lines for } t \geq 0]\) related to a unit length. We assume ideal contact between the surfaces of the band and the half-space. We choose an orthogonal coordinate system \( Oxyz \). The contact plane \( y = 0 \) coincides with the interface between the band and the half-space; the half-space occupies the region \( y \leq 0 \). The \( z \) axis is directed along the band's generatrix.

The problem is reduced to the solution of the differential equation

\[
\Delta w = \frac{1}{c^2} \frac{\partial^2 w}{\partial t^2} \left( c^2 = \frac{G}{\rho}, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \tag{1.1}
\]

which results from the Lamé equations in the absence of mass forces under the boundary conditions (1.2)

\[
y = 0: \quad w = \gamma f(t) \quad (|x| \leq a), \quad \tau_{yz} = 0 \quad (|x| > a) \tag{1.2}
\]

and the initial conditions

\[
t = 0: \quad w = 0, \quad \partial w / \partial t = 0. \tag{1.3}
\]

Here \( w(x, y, t) \) is the projection of the displacement vector onto the \( z \) axis, \( \rho \) and \( G \) are the density and shear modulus of the material of the elastic half-space, \( \tau_{yz} \) is the tangential component of the stress tensor, and \( \gamma f(t) \) is a function that characterizes the rigid displacement of the band.

We shall solve the mixed boundary-value problem (1.1)–(1.3) using integral transforms [3]. Applying
the Laplace–Carson transform with respect to time

$$w^L = p \int_0^\infty w(x, y, t)e^{-pt}dt, \quad w = \frac{1}{2\pi i} \int w^L(x, y, p) \frac{e^{pt}}{p} dp,$$

(1.4)

we obtain the following boundary-value problem for the function $w^L(x, y, p)$:

$$\Delta w^L = p^2c_2^2w^L, \quad y = 0: \quad w^L = \gamma f^L(p) \quad (|x| \leq a), \quad \tau^L_{yz} = 0 \quad (|x| > a).$$

(1.5)

To solve (1.5), we use the integral Fourier transform with respect to $x$ [1] and write an integral equation with respect to the Laplace–Carson-transformed contact shear stresses $\tau^L(x, p)$. Using the dimensionless variables $x = x'a$ and $\xi = \xi'a$ and the notation $\gamma = \gamma'a$, $\lambda = c_2(ap)^{-1}$, and $\varphi^L(z', p) = \tau^L(z, p)G^{-1}$ (the prime is further omitted), we write this equation as

$$k(s) = \int K(u) \cos(us) du = K_0(s), \quad K(u) = \frac{1}{\sqrt{1 + u^2}}, \quad g(x) \equiv \gamma,$$

(1.7)

where $K_0(s)$ is the MacDonald function.

We consider the second contact problem on frictionless pressing of a rigid punch of width $2a$ in the elastic half-plane $|z| < \infty, x \leq 0$ by the force $P(t) = P_0f(t)$, i.e., it is required to find a solution of the Lamé system of equations, which is written for convenience in terms of the wave functions $\varphi(x, y, t)$ and $\psi(x, y, t)$ as

$$\frac{\partial}{\partial x} \left( c_1^2 \Delta \varphi - \frac{\partial^2 \varphi}{\partial y^2} \right) + \frac{\partial}{\partial y} \left( c_2^2 \Delta \psi - \frac{\partial^2 \psi}{\partial x^2} \right) = 0,$$

$$\frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} = 0 \quad \left( b = \frac{c_2}{c_1}, \quad |x| > a \right),$$

(1.8)

and is subject to the boundary and initial conditions

$$y = 0: \quad 2 \frac{\partial \varphi}{\partial x} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad (|x| < \infty),$$

$$\frac{1}{b^2} \frac{\partial^2 \varphi}{\partial y^2} + \left( \frac{1}{b^2} - 2 \right) \frac{\partial^2 \varphi}{\partial x^2} + 2 \frac{\partial^2 \psi}{\partial x \partial y} = 0 \quad \left( b = \frac{c_2}{c_1}, \quad |x| > a \right),$$

$$\frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x} = -[\gamma - r(x)]f(t) \quad (|x| \leq a);$$

$$t = 0: \quad \varphi = \psi = 0, \quad \frac{\partial \varphi}{\partial t} = \frac{\partial \psi}{\partial t} = 0.$$

(1.9)

In (1.8) and (1.9), $\nu$ is the Poisson ratio for the material of the elastic half-plane, $\gamma f(t)$ is the rigid displacement of the punch under the action of the force $P(t)$, and $r(x)$ is a function that describes the punch-base shape. Here we confine ourselves to the case in which the force $P(t)$ is applied at the center of the punch collinearly to the $y$ axis.

To (1.8)–(1.10) we apply in sequence the Laplace–Carson integral transform with respect to time and the Fourier transform with respect to the $x$ coordinate. Thus, we reduce, as before, the solution of the mixed boundary-value problem to the equivalent integral equation in the Laplace–Carson images. This equation in the above dimensionless variables takes the form (1.6). The kernel of the equation is representable in the form (1.7), where

$$K(u) = \frac{2(1 - b^2)\sqrt{u^2 + b^2}}{(2u^2 + 1)^2 - 4u^2\sqrt{u^2 + b^2} + b^2\sqrt{u^2 + 1}} \quad (b^2 < 1).$$

(1.11)