EVOLUTION EQUATION FOR WEAKLY NONLINEAR WAVES IN A TWO-LAYER FLUID WITH GENTLY SLOPING BOTTOM AND LID

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A second-order differential model for three-dimensional perturbations of the interface of two fluids of different density is constructed. An evolution equation for traveling quasistationary waves of arbitrary length and small but finite amplitude is obtained. In the case of the horizontal bottom and lid, there are perturbations of the Stokes-wave type among steady-state periodic solutions. For moderately long perturbations, solutions in the form of solitary waves which are in agreement with the available experimental and analytical results are found. The problem of a smooth transition from the deep-fluid to the shallow-fluid region is studied.

Borisov and Khabakhpashev proposed a very simple differential model capable of describing the dynamics of long and short three-dimensional, weakly nonlinear perturbations of the interface of two fluids of different density confined by a rigid horizontal bottom and lid. However, the derivation of the wave-type equation for quasistationary disturbances was not quite correct. In addition, formally, even a linearized equation can have unstable solutions. The purpose of this work is to obtain a second-order differential model and a corresponding evolution equation that is free from the above-mentioned disadvantages without requiring the layers to be of constant depth.

1. Second-Order Differential Model. It is assumed that the fluids are ideal, incompressible, and immiscible, the stationary components of the fluid motion equal zero, the occurring oscillating flows are potential, and the waves are weakly nonlinear (i.e., \( \eta_a k/\tanh(kh_m) \sim \varepsilon \), where \( \eta_a \) is the amplitude of the disturbance at the interface, \( k \) is the wave number, \( h_m \) is the depth of the smaller layer, and \( \varepsilon \) is a small parameter). Third-order infinitesimals are omitted with capillary effects ignored.

In [1], the initial system of hydrodynamic equation was reduced to the equations

\[
\frac{\partial \eta}{\partial t} + \nabla \cdot \{(u_i)[\eta + (-1)^l \xi_i]\} = 0, \quad (1.1)
\]

\[
\frac{\partial u_{li}}{\partial t} + \nabla \left( g \eta + \frac{u_{li}^2}{2} + \frac{p_i}{\rho_i} \right) + \frac{\partial^2 \eta}{\partial t^2} \nabla \eta = 0 \quad (1.2)
\]

by integrating over the vertical coordinate and by using the standard kinematic and dynamic boundary conditions on the lid, the bottom, and the interface. Here \( t \) is the time, \( u \) is the vector of the horizontal component of the fluid velocity, the angular brackets indicate its value averaged over the layer depth, \( g \) is the acceleration of gravity, \( \rho \) is the density, \( p \) is the pressure, \( l = 1 \) for the upper fluid, and \( l = 2 \) for the lower one; the subscript \( i \) indicates the values of the quantities related to the interface and the gradient operator \( \nabla \) is determined in the horizontal plane.

Then, in [1], the use of well-known dependences for the vertical profiles \( u_i \) (see, for example, [2]) enabled one to relate the Fourier components of the boundary and averaged velocities of the fluids:

\[
u_{li}(\omega, k) = kh_l \coth(kh_l)\{u_i(\omega, k)\}, \quad (1.3)\]
Fig. 1. The dimensionless phase velocity $c_*$ as a function of the dimensionless wave number $k_*$ for $h_2/h_1 = 3$ and $p_2/p_1 = 1.25$: curves 1 and 2 refer to calculation by the approximation (1.5) for $\alpha = 0$ and $2/3$, respectively, and curve 3 to calculation by the exact dispersion relation (1.4).

where $\omega$ is the cyclic frequency. If $\nabla h \sim 3^{1/2}$, the formulas for waves in a liquid of constant depth are also locally true for a weakly inclined bottom and lid. In this connection, we give one more classical formula, namely, the dispersion relation for linear monoharmonic vortex-free waves in a two-layer liquid [2]:

$$\omega^2 [\rho_1 \coth(kh_1) + \rho_2 \coth(kh_2)] = gk(\rho_2 - \rho_1). \tag{1.4}$$

We replace approximately the transcendental equation (for the wave number) by the following simplest Padé approximation:

$$\omega^2(1/A_\omega + \omega_*^2) = \omega^2/c_*^2 = k^2 c_0^2 = k^2 g \delta, \quad A_\omega = 1 + \alpha \omega_*^2, \quad \omega_*^2 = \omega^2 \beta/g_+, \tag{1.5}$$

$$g_+ = g \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}, \quad \beta = h_1 h_2 \frac{\rho_1 + \rho_2}{\chi}, \quad \delta = h_1 h_2 \frac{\rho_2 - \rho_1}{\chi}, \quad \chi = \rho_1 h_2 + \rho_2 h_1.$$

Here $c_*$ is the phase velocity (the subscript 0 indicates its value for waves of infinitely small frequency) and $\alpha$ is the numerical coefficient. If $\alpha = 0$, we have the simplest polynomial approximation suggested in [1]. In this case, the long-wave ($\omega_*^2 \ll 1$) and short-wave ($\omega_*^2 \gg 1$) asymptotic curves coincide with the exact dispersion curve. In addition, in the range of intermediate frequencies the approximation error is determined by the ratios of the layer depths and the fluid densities. In particular, for $h_2/h_1 = 3$ and $p_2/p_1 = 1.25$ (the experiment in [3] was performed for these values), the maximum relative deviation of the exact relation (1.4) from the approximate (1.5) with $\alpha = 0$ is 8.5%, and it is reached for $k_* = kH \approx 4$ ($H = h_1 + h_2$ is the distance between the bottom and the lid). If $\alpha = 2/3$, the approximation error does not exceed 2%, and the corresponding maximum is attained for $k_* \approx 6$ (Fig. 1). A comparison of relations (1.4) and (1.5) leads to the expressions

$$kh_1 \coth(kh_1) = 1/A_\omega + \omega_*^2 h_1/g_+^2 \text{ by means of which we write Eqs. (1.3) in the form}$$

$$A_\omega u_{il}(\omega, k) = (1 + A_\omega \omega_*^2 h_1/g_+^2)u_l(\omega, k). \tag{1.6}$$

Application of the inverse Fourier transform to (1.6) yields the differential relations for the boundary and averaged velocities

$$A_t u_{li} = (u_l) - A_t \frac{h_1}{g_+} \frac{\partial^2(u_l)}{\partial t^2}, \quad A_t = 1 - \alpha \frac{\beta}{g_+} \frac{\partial^2}{\partial t^2}. \tag{1.7}$$

In contrast to formulas (1.7), relations (1.3) and (1.6) also remain valid in the case of weakly nonlinear perturbations. Therefore, we can use the generalization of expressions (1.7)

$$A_t u_{li} = (u_l) - A_t \left\{ \frac{h_1}{g_+} + \frac{(-1)^l}{g_+} \left( \frac{3 h_1}{g_+} \frac{\partial^2}{\partial t^2} \right) \frac{\partial^2(u_l)}{\partial t^2} - 4u_{li}' \right\}, \tag{1.8}$$