INTEGRAL EQUATION FOR STRESS CONCENTRATION
AT THE EDGE OF A PLANE CRACK OF ARBITRARY Contour

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Equations are derived for stress concentration near a crack of closed contour lying in a plane. A system of one-dimensional integral equations for the concentration factor is obtained. The right sides of the equations contain the initial approximation—a solution of the problem of a circular crack whose sides are acted upon by nonaxisymmetric loading.

In solving mixed problems for harmonic functions, it is necessary to evaluate functions on boundary segments on which their values are not specified in boundary-value problems. For example, in the problem of stationary filtration of a liquid by the Darcy rule into the depth of a homogeneous porous half-space through a permeable spot on the surface, the pressure of the overlying liquid on the spot is known, and in the impermeable part of the boundary outside the permeable spot, the normal component of the vector velocity is equal to zero. A calculation of the liquid velocity normal to the permeable part of the boundary is required to determine the liquid flow rate. In the mixed problem of a brittle opening-mode crack, displacements on the crack extension in the crack plane and varying stress normal to the crack are specified. An interesting quantity in this problem is the stress on the crack extension—since from the stress-intensity factor, it is possible to determine the stable form of the crack.

Usually, the boundary equations of potential theory are employed in determining such quantities. In the crack problem, two methods of calculation are possible. The first method uses the Fredholm equation of the first kind, in which the displacement \( w \) on the crack is expressed in terms of the stress \( \sigma \) as

\[
\sigma(r, \vartheta, 0) = \frac{1}{2\pi A} \int_0^\infty \int_0^{2\pi} R(p, r, \vartheta - \alpha) \rho \sigma(p, \alpha, 0) \frac{dp\, d\alpha}{R(p, r, \vartheta - \alpha)}. \tag{1}
\]

Here and below, \( z = 0, r < l(\vartheta) \) is the position of the crack in the cylindrical coordinates \( (r, \vartheta, z) \), \( R(p, r, \vartheta - \alpha) \) is the distance between the points \( (r, \vartheta) \) and \( (p, \alpha) \), and \( A = \mu/(1 - \nu) \), where \( \mu \) and \( \nu \) are the shear modulus and Poisson's constant.

Since, by virtue of symmetry, the displacement is equal to zero on the crack extension \( z = 0 \) and \( r > l(\vartheta) \), we obtain an equation for the unknown function \( \sigma = \sigma_+ \) at \( z = 0 \) and \( r > l(\vartheta) \) [the values of \( \sigma = \sigma_- \) for \( r < l(\vartheta) \) are specified]. The instability of calculation schemes for equations of this type and the unboundedness of the region in which solutions are sought hinder the search for singular solutions, and the sought function is a singular solution.

In the second method, an equation for the boundary solution is obtained from the equation given above by inversion if the integrals are understood as an integral transformation of the function \( \sigma \) to \( w \):

\[
\sigma(r, \vartheta, 0) = A \frac{1}{2\pi} \lim_{\varepsilon \to 0} \varepsilon \int_0^{2\pi} \int_0^\infty \frac{dw(p, \alpha, 0)}{\varepsilon^2 + R^2(p, r, \vartheta - \alpha)^{3/2}}. \tag{2}
\]

The parameter \( \varepsilon \) is introduced to reduce the singularity of the kernel.
For \( r < l(\vartheta) \), this expression is an integral equation for displacement. (It contains the derivative of the double-layer potential with respect to the normal.) It is stable in calculations, but the high degree of singularity leads to difficulties in numerical implementation. In addition, the stress is evaluated from the displacement found, and this introduces an additional error in calculations of the stress-intensity factor.

In the present paper, to calculate the stress-intensity factor, we derive a modified integral boundary equation that is "intermediate" between Eqs. (1) and (2).

By means of Papkovich representations, determination of the opening-mode crack parameters reduces to seeking one harmonic function \( f(r, \vartheta, z) \). The displacements and stresses can be expressed in terms of this function. For example, for the normal components of the displacement and stress vectors on a site with normal parallel to the \( z \) axis, we have

\[
\begin{align*}
w &= 2\left(1 - \nu - \frac{z}{2} \frac{\partial f}{\partial z}\right) \frac{\partial f}{\partial z}, \\
\sigma &= 2\mu \left(1 - \frac{z}{2} \frac{\partial^2 f}{\partial z^2}\right).
\end{align*}
\]

In the problem of an opening-mode crack whose points \( r < l(\vartheta) \) lie in the plane \( z = 0 \), the function \( f \) should satisfy the conditions

\[
\frac{\partial^2 f}{\partial z^2} = \frac{1}{2\mu} \sigma(r, \vartheta) \quad [r < l(\vartheta)], \\
\frac{\partial f}{\partial z} = 0 \quad [r > l(\vartheta)].
\]

It is assumed that the crack contour does not have angular points, and, for simplicity, it is considered star-shaped.

We examine the upper half-space. Expanding the functions \( f, w, \) and \( \sigma \) in complex Fourier series in the angular coordinate and performing a Hankel transformation with kernel \( r J_n(qr) \) (\( q \) is a transformation parameter) for the factors \( w_n \) and \( \sigma_n \), for each harmonics we obtain solutions that decrease exponentially along \( z \). In the crack plane, the Hankel images of the Fourier coefficients are related by the condition (for \( z = 0 \), the dependence on this argument for all functions is omitted below)

\[
\sigma_n^H(q) = -A q w_n^H(q).
\]

In this equality, we convert to preimages. We invert the Hankel images with integer indices using the inversion formula for the Hankel transformation with half-integer indices. Multiplying the last equality by \( \sqrt{q} J_{n+1/2}(qx) \) and integrating it with respect to \( q \) taking into account the formula for discontinuous integrals (see [1, formula 6.575.1]), we have

\[
\int_0^\infty q^{\mu-n} J_{\nu+1}(aq) J_{\mu}(bq) dq = \begin{cases} 0, & a < b, \\
\frac{(a^2 - b^2)^{\nu-\mu} a^{\mu}}{2\nu-\mu a^{\nu+1} \Gamma(\nu - \mu + 1)}, & a > b,
\end{cases}
\]

where \( \Gamma \) is a gamma function.

We obtain the following equations for the coefficients of the harmonics:

\[
\int_0^\infty \frac{\rho \sigma_n(\rho)}{\sqrt{x^2 - \rho^2}} (\frac{\rho}{x})^n d\rho = A x \int_0^\infty \frac{\partial}{\partial \rho} \left[w_n(\rho)(\frac{\rho}{x})^n\right] \frac{d\rho}{\sqrt{\rho^2 - x^2}}.
\]

(3)

After summation over \( n \) in infinite limits with weight \( \exp(in\vartheta) \) taking into account the expressions of the Fourier coefficients in terms of the functions expanded in a series, the equations reduce to the following integral equation with respect to the unknown half-opening of the crack \( w \) and the stress on the crack extension \( \sigma = \sigma_+ + \sigma_+ \), where \( \sigma_- \) is specified:

\[
\int_0^{2\pi} \frac{\rho \sigma(\rho, \alpha) \sqrt{x^2 - \rho^2} d\rho d\alpha}{R^2(\rho, x, \vartheta - \alpha)} = A x \int_0^{2\pi} \frac{\partial}{\partial \rho} \left[w(\rho, \alpha)(\rho^2 - x^2)\right] \frac{d\rho d\alpha}{\sqrt{\rho^2 - x^2}}.
\]

(4)

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