APPLICATION OF PERTURBATION THEORY TO THE
NONLINEAR VOLTERA–GAUSE–WITT MODEL FOR
PREY–PREDATOR INTERACTION

Ranabir Dutt, P. K. Ghosh and B. B. Karmakar
Department of Physics, Visva-Bharati,
Santiniketan-731235, W. Bengal,
India

Krylov–Bogoliubov–Mitropolsky perturbation method was used to study the effect of nonlinearity in the Volterra–Gause–Witt (VGW) model for a two species prey–predator system. The first order corrections to both the frequency of oscillation and the amplitude of the linearized system were computed. It was found that the basic qualitative features of the nonlinearity are exhibited by the first order result. We have also discussed the Lotka–Volterra problem which is a special case of VGW model.

1. Introduction. One of the most interesting observations in the study of population dynamics is that the populations of a pair of species bound by the prey–predator relationship, fluctuate periodically. The mathematical investigation of this problem was initiated by Lotka and Volterra (Goel and Maitra, 1971; Bartlett, 1960; Gause, 1971) who proposed a set of rate equations with quadratic nonlinearities which describe phenomenologically the interaction of a prey species with its predator when both populations coexist in an ecological niche with finite resources. It is assumed that the prey (Species 1) would grow exponentially in the absence of the predator (Species 2) while the predator dies out exponentially in the absence of its prey. The Lotka–Volterra equations are

\[
\begin{align*}
\frac{dN_1}{dt} &= \alpha_1 N_1 - \beta_1 N_1 N_2 \\
\frac{dN_2}{dt} &= -\alpha_2 N_2 + \beta_2 N_1 N_2
\end{align*}
\]
where $N_i$ ($i = 1, 2$) is the number of individuals of the species $i$ at a given time, $\alpha_i$ is the innate capacity for increase per individual (intra-specific parameter) and $\beta_i$ is the coefficient of mutual interaction (interspecific or niche overlap parameter). In this form of the equations describing a prey–predator system, these coefficients are all positive.

A deficiency of the Lotka–Volterra (LV) model is the non-existence of a saturation level for the population of the prey species alone (in the absence of the predator) which, in general does not grow indefinitely in a given environment with limited space and resource. The saturation effect was later incorporated by Gause and Witt who introduced a self-interaction term in the Verhulst form. The modified Volterra–Gause–Witt (VGW) model (Gause and Witt, 1935) is described by the instantaneous growth equations

$$\begin{align*}
\frac{dN_1}{dt} &= \alpha_1 N_1 (1 - N_1/\theta) - \beta_1 N_1 N_2 \\
\frac{dN_2}{dt} &= -\alpha_2 N_2 + \beta_2 N_1 N_2
\end{align*}$$

(2)

where $\theta$ is the carrying capacity (self saturation level) of Species 1.

It is clear that the VGW model reproduces the LV system in the limit $\theta \to \infty$.

It is quite difficult to handle these nonlinear population models analytically because the exact solutions have not been obtained so far. In the case of the LV model there is an advantage: the equations (1) describe a conservative system and correspond to a simple closed loop in the $(N_1N_2)$ phase plane. This certainly indicates that periodic solutions exist for $N_1$ and $N_2$ with a single unique frequency. However, it is not at all clear whether the VGW model will simulate similar periodic fluctuations in these variables. When the exact solutions are not available, the standard procedure generally followed, is to linearize the nonlinear equations in the neighborhood of the equilibrium point in the phase plane, assuming a priori that the effect of the nonlinear terms would be small (Rosen, 1970; Minorsky, 1962). Essentially, this assumption motivates us to work out a perturbative calculation to determine the weak non-linear effects.

In Section 2, we shall give an outline of the prescription for linearization (Rosen, 1970) and obtain the analytic form of the amplitude and frequency of oscillation. In Section 3, we shall discuss how the asymptotic perturbation theory of Krylov–Bogoliubov–Mitropolsky (KBM, cf. Minorsky, 1962) may be applied to our problem to estimate the first order correction, due to non-linearity, to the linear frequency and to the amplitude. Some remarks on the first order results will be given in the concluding section.

2. Linearization of the Problem. As a guideline, we shall utilize the lineariza-