LIMIT DISTRIBUTIONS OF FINITE MARKOV CHAINS AND SEMI-GROUPS OF STOCHASTIC MATRICES

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INTRODUCTION

We recall one theorem from the theory of summation of independent random variables ([1, Ch. IX, Thm. 2]).

THEOREM. The following classes of probability distributions coincide:
(i) limit distributions of sums

\[ S_{N,n(N)} = \sum_{i=1}^{n(N)} X_{N_i}, \]

where in each series random variables \(X_{N_i} | 1 \leq i \leq n(N)\) are independent and identically distributed and a sequence of integer numbers \(n(N)\) tends to \(+\infty\);
(ii) distributions of increments of stochastic processes with independent and stationary increments;
(iii) infinitely divisible distributions.

In the present note we obtain the similar characterization of limit distributions of \(S_{N,n(N)}\) in the case where \((S_{N,i} | i \geq 0)\) is a finite Markov chain.

The connecting link between the theory of summation of independent random variables and theory of Markov chains is an operator approach in the theory of summation ([1, Chs. VIII, IX]). Let \(C\) denote the linear space of continuous functions on the real line \(\mathbb{R}\) with finite limits at infinity and with supremum norm. For any probability \(F\) on \(\mathbb{R}\) we define the linear operator \(\tilde{F}\) on \(C\) by the formula

\[ (\tilde{F} f)(x) = \int f(x - y)F(dy). \]

Then the following three classes of operators correspond to the classes (i), (ii), and (iii) of distributions, respectively:

1) class of limits in strong operator topology of operators of the form \(\tilde{F}_{n(N)}\);
2) operators of the form \(\tilde{F}(1)\), where the family \((\tilde{F}(u) | u \geq 0)\) is continuous with respect to \(u\) in the strong operator topology and satisfies the identity \(\tilde{F}(u + v) = \tilde{F}(u)\tilde{F}(v)\);
3) class of operators \(\tilde{F}\), for any \(k \geq 1\) admitting the representation \(\tilde{F} = \tilde{F}_k\) with some operator \(\tilde{F}_k\).

Now let \((S_n | n \geq 0)\) be, finite homogeneous Markov chain with a matrix of transition probabilities \(P\). If \(p^T\) is row vector of initial probabilities of the chain then the distribution of the random variable \(S_n\) is given by the row \(p^T P^n\). Therefore in order to find the limit distributions of \(S_n\) it is sufficient to investigate the asymptotic of the sequence of matrices \(P^n\).

Before the formulation of the main results of the note we shall recall some definitions. A stochastic matrix \(J\) is called quasi-projector if \(J^{s+1} = J\) for some \(s \geq 1\).
A stochastic matrix $L$ is called infinitely divisible if for any $k \geq 1$ it can be represented in the form $L = L_k^k$ with some stochastic matrix $L_k$.

By continuous semi-group of stochastic matrices we understand a family $(L(u) \mid u > 0)$ of stochastic matrices continuous with respect to $u$ and satisfying the semi-group identity

$$L(u + v) = L(u)L(v).$$  \hfill (1)

**THEOREM 1.** The following classes of stochastic matrices coincide:

(i) limit matrices for $P_n^{(N)}$, where $P_N$ is a sequence of stochastic matrices and $n(N) \to +\infty$;

(ii) matrices of the form $L(1)J$, where $L(u) \mid u > 0$ is a continuous semi-group of stochastic matrices and $J$ is a quasi-projector commuting with any $L(u)$;

(iii) matrices of the form $\exp(-\mu)\exp(\mu \Pi)J$, where $\mu > 0$, $\Pi$ is a stochastic matrix and $J$ is a quasi-projector commuting with $\Pi$.

More over, any infinitely divisible matrix $L$ is of the form

$$L = \exp(-\mu)\exp(\mu \Pi)J,$$

where $\mu > 0$, $\Pi$ is a stochastic matrix and $J$ is a quasi-projector commuting with $\Pi$.

The proof of this theorem is based on the following result which is of independent interest itself.

**THEOREM 0.** Let $P_N$ be a sequence of stochastic matrices and $n(N) \to +\infty$. Then there exist $s \geq 1$ and a subsequence $(N') \subset (N)$ such that for any sequence of integers $m(N')$ satisfying the conditions

$$m(N')/n(N') \to u \in [0; +\infty[,$$

there exists a limit $\lim P_{N'}^{m(N')}$. If $P_N \to \overline{P}$, then as $s$ we can take a number of cyclic classes of the matrix $\overline{P}$.

This theorem can be reformulated as follows.

**THEOREM 2.** Let $P_N$ be a sequence of stochastic matrices converging to acyclic stochastic matrix, $n(N) \to +\infty$ and

$$Q_N(u) = P_n^{[\alpha(u)}$$

where $[\alpha]$ stands for the integer part of $\alpha$. Then there exists a continuous semi-group of stochastic matrices $L(u)$ and a subsequence $(N') \subset (N)$ such that $Q_N(u) \to L(u)$ uniformly on compact subsets of the interval $[0; +\infty[$.

1. **PROOFS**

First we recall the main notation which will be used in the proofs. The notation arg $z$ and log $z$ stand for the principal value of argument and the logarithm of a complex number $z$, $t$ denotes the imaginary unit and $[x]$ denotes the integer part of a real number $x$.

By $P$ we will denote stochastic matrices, $\sigma(P)$ denotes the spectrum of a matrix $P$ and $\sigma^*(P)$ denotes the part of the spectrum consisting of eigenvalues of modulus 1. A matrix is called a simple one if all its eigenvalues are distinct. $\| \cdot \|$ denotes any matrix norm.

Practically all variables which will be used in the proofs (only with exception for $P$, $\mu$, and $L$) will denote sequences of objects, but for brevity we shall omit the index $N$. Any statement on the convergence of any sequence to any limit must be understood as the convergence as $N \to \infty$, if not stated in the contrary.

Before the proof of Theorem 0 we provide the following two lemmas.

**LEMMA 1.** Let $P \to \overline{P}$, $n \to +\infty$, $\sigma^*(\overline{P}) = \{1\}$, $\lambda \in \sigma(P)$, $\liminf |\lambda|^n > 0$. Then $n \arg \lambda = O(1)$.

**Proof.** Suppose the contrary and take a sequence such that $\lambda^n \to \mu \neq 0$ and $n|\arg \lambda| \to +\infty$. Set $m = [1/|\arg \lambda|]$.

Since $|\lambda|^m \to |\mu| \neq 0$, then $|\lambda| \to 1$; therefore $\lambda \to 1$, since $\sigma^*(\overline{P}) = \{1\}$. Thus, $\arg \lambda \to 0$ and this implies $m \arg \lambda \to 1$. Whence $m/n \to 0$, therefore $|\lambda|^m \to 1$ and $\lambda^m \to \exp(i)$. We arrived at a contradiction, since the number $\exp(i)$ can not be an eigenvalue of any stochastic matrix. The lemma is proved.