ON CHAINS OF RELATED SETS

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If $K$ is a partition of a set $K$ which is partially ordered by the relation $R$ and $R$ is a collection of pairs of sets of $K$ such that the sets of each pair are related by $R$ in the sense of Rashevsky, then $R$ is a relation which partially orders $K$. Necessary and sufficient conditions that $K$ be a chain are obtained, and if $K$ is a chain under these conditions, it is shown that $K$ is unique.

Certain propositions on relations between sets were presented by N. Rashevsky (1961) and M. Sommerfield (1963). In this paper $K$ denotes a finite set which is partially ordered by a relation denoted by $R$, i.e. $R \subseteq K \times K$ which is reflexive, antisymmetric and transitive. We investigate collections of subsets of $K$ such that if $A, B$ is a pair of subsets of $K$, then $A$ and $B$ are related by $R$ in the sense of Rashevsky, and state conditions under which such a collection forms a chain.

For convenience we restate the definitions (Rashevsky, 1961) of relations between sets. The statement that the set $A$ is strongly related by $R$ to the set $B$, symbolized by $ARB$, means that $A \times B \subseteq R$. The statement that $A$ is weakly related by $R$ to $B$, symbolized by $AR'B$, means that $A \times B \notin R$ and if $R_{AB} = A \times B \cap R$, then $R_{AB}$ is from $A$ onto $B$. We use the symbol $ARB$ to signify $ARB$ or $AR'B$.

The following theorem is a slight extension of Theorem 13 (Sommerfield, 1963).

**Theorem 1:** If each of $A$, $B$ and $C$ is a set of a collection of subsets of $K$, $ARB$ and $BRC$, then $ARC$. Furthermore, if $ARB$ or $BRC$, then $ARC$.  

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Proof: If $x \in A$, there is a $y \in B$ such that $(x, y) \in R$, and if $y \in B$, there is a $z \in C$ such that $(y, z) \in R$. Hence $(x, z) \in R$. Similarly, if $v \in C$, there is a $u \in A$ such that $(u, v) \in R$. Hence $ARC$. Suppose $ARB$ or $BRC$. Then if $v \in C$ and if $u \in A$, $(u, v) \in R$ so that $ARC$.

We observe that $ARB$ implies that not more than two elements of $K$ are common to $A$ and $B$. Hence $ARB$ does not imply $ARA$. Since the remainder of this paper is concerned with collections of subsets of $K$ no two of which contain a common element, it may happen, for example, that if $A$, $B$ and $C$ are subsets of $K$, then $ARB$, $ARC$, $AR'C$ and $BRB$.

The statement that $K$ is a partition of $K$ means that $K$ is a collection of subsets of $K$ no two of which intersect such that $\cup_{A \in K} A = K$. With $R$ denoting the collection of pairs $(A, B)$ of sets of $K$ such that $ARB$, the following theorem shows that the relation $R$ induces a partial ordering of $K$ by $R$.

**Theorem 2:** If $K$ is a partition of $K$, then $K$ is partially ordered by $R$.

**Proof:** If $A \in K$ and $x \in A$, then $(x, x) \in R_A$, so that $(A, A) \in R$. Suppose $(A, B) \in R$ and $B \neq A$. If $x \in A$, there is a $y \in B$ such that $(x, y) \in R$. Suppose if $y \in B$, there is a $z \in A$ such that $(y, z) \in R$. Then if $x \in A$, there is a $z \in A$, $z \neq x$, such that $(x, z) \in R$. Since $A$ is finite, this leads to the contradiction of a pair $u, v$ of elements of $A$ such that $(u, v) \in R$ and $(v, u) \not\in R$. Hence there is a $y \in B$ such that if $x \in A$, then $(y, x) \not\in R$. Thus $(B, A) \not\in R$. It follows from Theorem 1 that $R$ is transitive.

Examples of finite partially ordered sets are conveniently given by means of diagrams. If $M$ is partially ordered by $S$, the statement that an element $q$ of $M$ covers the element $p$ of $M$ means that $(p, q) \in S$ and if $(p, z) \in S$, $z \neq p$ and $z \neq q$, then $(z, q) \not\in S$. A figure obtained by representing elements of $M$ by dots so that if $q$ covers $p$, then the dot for $q$ is above the dot for $p$, and connecting the dots for $p$ and $q$ with a line segment is called a diagram of $M$. According to a theorem of Birkhoff (1935) every finite partially ordered set is representable by a diagram. In Figure 1 are represented three partitions of a 7-element set $K$.

![Figures 1a, b, c.](image-url)