ECCENTRIC MULTipoLE REPRESENTATIONS OF CURRENT
GENERATORS IN A SPHERICAL VOLUME CONDUCTOR*

GORDON C. K. YEH**
CONSULTANT,
REED RESEARCH FOUNDATION
WASHINGTON, D. C.

In Yeh, Martinek and de Beaumont (Bull. Math. Biophysics, 20, 203-16, 1958), a method is presented for determining successively better central multipole representations of the current generators in a homogeneous conducting sphere by measuring surface potentials at a successively increasing number of points. This paper generalizes the method such that the multipoles may be located at any chosen point in the conductor. The spherical harmonic expansion is advantageously used and the "interior sphere theorem" of Ludford, Martinek and Yeh (Proc. Cambridge Philos. Soc., 51, 389-93, 1955) makes possible disturbance potential expressions in closed forms. A method for approximate determination of the eccentricity is also presented. In the theory of electrocardiography, the eccentric multipoles can more accurately represent the heart as a current generator with fewer surface potential measurements than the central multipoles.

Eccentric Multipole Expansion of Any Potential in an Infinite Domain. Consider two systems of coordinates: one with the origin at \( x = 0, y = 0 \) and \( z = 0 \) and the other with the origin at \( x = x' = 0, y = y' = 0 \) and \( z = z' + b = b \). The relations between the rectangular coordinates \((x, y, z)\) and the spherical coordinates \((r, \theta, \varphi)\) are

\[
\begin{align*}
x &= r \sin \theta \cos \varphi \\
y &= r \sin \theta \sin \varphi \\
s &= r \cos \theta
\end{align*}
\]

(1)

*This investigation was supported by The National Heart Institute under Research Grant H-2263(c-4).
**Member of the technical staff, Research Laboratory, Space Technology Laboratories, Inc., Canoga Park, California, and Consultant to Reed Research Foundation.
and those between \((x', y', z')\) and \((r', \theta', \varphi')\) are
\[
\begin{align*}
x' &= r' \sin \theta' \cos \varphi' = r' \sin \theta' \cos \varphi \\
y' &= r' \sin \theta' \sin \varphi' = r' \sin \theta' \sin \varphi \\
z' &= r' \cos \theta' = r' \cos \theta'
\end{align*}
\]
since \(\varphi' = \varphi\).

Let \(\rho(r_0, \theta_0, \varphi_0)\) be a current source density distribution which is zero outside the domain \(r < \alpha_0\). The potential at a point outside of this domain is given by the following series (see, for example, Morse and Feschbach, 1953, p. 1276):
\[
\phi'(r', \theta', \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[ A_{mn} \phi_{mn}^e(r', \theta', \varphi) + B_{mn} \phi_{mn}^o(r', \theta', \varphi) \right]
\]
where
\[
\begin{align*}
\phi_{mn}^e(r', \theta', \varphi) &= (r')^{-n-1} Y_{mn}^e(\theta', \varphi), \\
\phi_{mn}^o(r', \theta', \varphi) &= (r')^{-n-1} Y_{mn}^0(\theta', \varphi)
\end{align*}
\]
are the unit potentials for the even and odd \(mn\)th multipoles respectively and
\[
\begin{align*}
A_{mn} &= \frac{\varepsilon_m (n - m)!}{(n + m)!} \int_0^{2\pi} \cos (m\varphi) d\varphi \times \\
&\quad \int_0^{\pi} P_n^m(\cos \theta_0) \sin \theta_0 d\theta_0 \int_0^{\alpha_0} \rho(r_0, \theta_0, \varphi_0)(r_0)^{n+2} dr_0 \\
B_{mn} &= \frac{\varepsilon_m (n - m)!}{(n + m)!} \int_0^{2\pi} \sin (m\varphi) d\varphi \times \\
&\quad \int_0^{\pi} P_n^m(\cos \theta_0) \sin \theta_0 d\theta_0 \int_0^{\alpha_0} \rho(r_0, \theta_0, \varphi_0)(r_0)^{n+2} dr_0
\end{align*}
\]
are the strengths of the various multipoles \((B_{0n} = 0)\).

In equations (4),
\[
\begin{align*}
Y_{mn}^e(\theta', \varphi) &= \cos (m\varphi) P_n^m(\cos \theta'), \\
Y_{mn}^0(\theta', \varphi) &= \sin (m\varphi) P_n^m(\cos \theta'),
\end{align*}
\]