RE-ESTABLISHABILITY IN ABSTRACT BIOLOGY

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It is shown that any $(\mathcal{M}, \mathcal{R})$-system has some component which cannot be re-established after it has been inhibited. If there is only one such component, it must be central, that is, its inhibition stops the whole system. These results hold even when it is not assumed that $\mathcal{M}$ is connected.

The purpose of this note is to offer stronger versions of the theorems of R. Rosen (1958) about re-establishability and centrality of components of $(\mathcal{M}, \mathcal{R})$-systems.

A structure with two kinds of objects, components and repairers, and a feeding relation between them, is called an $(\mathcal{M}, \mathcal{R})$-system if it obeys the following axioms:

1. There is an exact matching between components $M_1, M_2, \ldots, M_n$ and repairers $R_1, R_2, \ldots, R_n$.
2. Each repairer $R_i$ feeds its corresponding component $M_i$, and it only.
3. Each repairer is fed by at least one component.

These structures, with their emphasis on input-output relations, were introduced by Rosen to serve as mathematical prototypes of metabolic systems. The axioms given here are actually a little simpler than his, but they suffice for seeing how to dispense with connectedness.

A component $M_i$ is called re-establishable if it does not feed its repairer $R_i$, even indirectly. To speak intuitively, the inhibition of such an $M_i$ is not final since $R_i$, being unaffected, can repair it.

For example, in Figure 1, $M_1, M_2$ and $M_3$ are re-establishable; but if an
arrow were drawn from $M_1$ to $R_3$, meaning $M_1$ feeds $R_3$, then none of the three would be re-establishable.

![Diagram of components](image)

Figure 1.

A component is said to be central if it feeds everything in the system, at least indirectly. That is, an inhibition of such a component stops the system, since every repairer and every component is affected.

In Figure 1, none of the components are central, but if $M_1$ feeds $R_3$, then each becomes central. (Once $M_1$ feeds $R_3$, the arrow below $R_3$ may be thrown away, thus closing the system.)

As Figure 1 suggests, these ideas may be phrased in terms of the theory of graphs, or networks. Then an $(\mathcal{M}, \mathcal{R})$-system becomes a special graph, a component node is re-establishable if there is no directed path from it to its repairer node, and a component node is central if every node may be reached from it by a directed path. But a knowledge of graph theory is not needed to read the following proofs.

**Theorem.** Any $(\mathcal{M}, \mathcal{R})$-system has at least one component which cannot be re-established.

To see this, choose a re-establishable component and relabel it $M_1$, if necessary. Its repairer $R_1$, also relabeled, is fed by some component other than $M_1$. (If $M_1$ feeds $R_1$ directly, it cannot be re-established.) Call this other component $M_2$ and its repairer $R_2$. If $M_2$ is re-establishable, the component feeding $R_2$ is again not $M_2$, and it cannot be $M_1$, since the feeding path $M_1 R_2 M_2 R_1$ would deny the re-establishability of $M_1$ (see Fig. 1). Thus, there is a third component $M_3$, feeding $R_2$, which, if re-establishable, leads to a fourth component, and so forth. The general step in this construction yields a component $M_k$, whose repairer $R_k$ is fed by some $M_{k+1}$ different from each of $M_1, \ldots, M_k$. For, if the component feeding $R_k$ were $M_i$, with $1 \leq i \leq k$, then the feeding path

$$M_i R_k M_k R_{k-1} M_{k-1} \ldots R_{i+1} M_{i+1} R_i$$