We show that when we represent ($\mathfrak{S}$, $\mathfrak{R}$)-systems with fixed genome as automata (sequential machines), we get automata with output-dependent states. This yields a short proof that ($\mathfrak{S}$, $\mathfrak{R}$)-systems form a subcategory of automata—and with more homomorphisms than previously exhibited. We show how ($\mathfrak{S}$, $\mathfrak{R}$)-systems with variable genetic structure may be represented as automata and use this embedding to set up a larger subcategory of the category of automata. An analogy with dynamical systems is briefly discussed. This paper presents a formal exploration and extension of some of the ideas presented by Rosen (Bull. Math. Biophysics, 26, 103-111, 1964; 28, 141-148; 28, 149-151). We refer the reader to these papers, and references cited therein, for a discussion of the relevance of this material to relational biology.

1. The category $M^1_k (\mathfrak{U})$. Let $\mathfrak{U}$ be a category whose objects are sets $A, B, \ldots$ and whose sets of maps, $H(A, B)$, with domain $A$ and range $B$, form subsets of the corresponding set $\hat{H}(A, B)$ of all set-theoretic mappings from $A$ into $B$. We assume henceforth that $\mathfrak{U}$ is closed under cartesian products, i.e., if $A, B, C, D$ are objects of $\mathfrak{U}$, $f \in H(A, B)$, and $g \in H(C, D)$ are maps of $\mathfrak{U}$, then $A \times C$ and $B \times D$ are objects of $\mathfrak{U}$, and $f \times g \in H(A \times C, B \times D)$, where $(f \times g)(a, c) = [f(a), g(c)]$.

We shall state certain other conditions necessary for $\mathfrak{U}$ as we go along. An interesting question—not studied here—is to give the weakest possible conditions on a category in order that it satisfy all our needs. Any category satisfying these conditions might then be called an ($\mathfrak{S}$, $\mathfrak{R}$)-category. An experimental question is then to discover which categories occurring in nature are ($\mathfrak{S}$, $\mathfrak{R}$)-categories.
A simple \((\mathfrak{M}, \mathfrak{R})\)-system on the category \(\mathfrak{X}\) is a quadruple

\[ K = (A, B, f, \Phi), \]

where \(A\) and \(B\) are objects of \(\mathfrak{X}\) such that \(H(A, B)\) is an object of \(\mathfrak{X}, f \in H(A, B),\) and \(\Phi \in H[B, H(A, B)].\)

\(\Phi\) is interpreted as the "genome" of the system, \(f\) as the initial metabolic component. In environment \(a \in A,\) the metabolic component \(g \in H(A, B)\) changes to \(\Phi[g(a)]\).

A sequential machine on the category \(\mathfrak{X}\) consists of a sextuple

\[ \Lambda = (S, M, N, s_0, \delta, \lambda), \]

where

- \(S,\) the set of states, is an object of \(\mathfrak{X};\)
- \(M,\) the set of inputs, is an object of \(\mathfrak{X};\)
- \(N,\) the set of outputs, is an object of \(\mathfrak{X};\)
- \(s_0 \in S;\)
- \(\delta \in H(S \times M, S),\) the next-state function; and
- \(\lambda \in H(S \times M, N),\) the next-output function*

\(s_0\) is the initial state of the system. If \(\Lambda\) receives input \(m\) when in state \(s,\) it will change to state \(\delta(s, m)\) and emit output \(\lambda(s, m)\).

Rosen (1964) has observed that we may represent the simple \((\mathfrak{M}, \mathfrak{R})\)-system \(K\) as a sequential machine \(\Lambda\) simply by setting

\[ S = H(A, B), M = A, N = B, s_0 = f; \]

so that

\[ \delta(f', a) = \Phi[f'(a)]; \]

\[ \lambda(f', a) = f'(a). \]

This, of course, requires that \(H(A, B)\) be an object of \(\mathfrak{X}.\) We may now give the characterization:

**Theorem 1:** A sequential machine \(\Lambda\) on the category \(\mathfrak{X}\) represents a simple \((\mathfrak{M}, \mathfrak{R})\)-system iff it has an output-dependent state function, i.e., iff \(\beta \in H(B, S)\) such that

\[ \delta(s, m) = \beta[\lambda(s, m)]. \]

**Proof:** Necessity follows from Rosen’s construction.

For sufficiency, suppose \(\Lambda = (S, M, N, s_0, \delta, \lambda)\) has \(\delta(s, m) = \beta[\lambda(s, m)].\) Then \(\Lambda\) represents the \((\mathfrak{M}, \mathfrak{R})\)-system \((A, B, f, \Phi)\) if we embed \(A \subseteq H(M, N)\) by \(s \rightarrow \lambda(s, .),\) and set

\[ A = M, B = N, f = s_0 \quad \text{and} \quad \Phi = \beta, \]

* I have followed Rosen's notation here. However, there is no standard notation, and elsewhere (e.g., Arbib, 1965) I have used \(\lambda\) and \(\delta\) in reverse roles.