SOME GENERAL THEOREMS ON THE MOTION OF
INCOMPRESSIBLE VISCOUS FLUIDS

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Some standard theorems about the motion of single fluids are reviewed and extended to the case of several fluids moving through each other. Some further results are obtained which do not have a counterpart in the case of a single fluid.

The mechanics of viscous fluids is of importance in the study of biological movements, diffusion processes, etc. The purpose of the present paper is to collect together some results of a more or less general nature, without entering into any detail as to actual solution of the equations of motion.

Single Fluid

1. An incompressible fluid* can move without viscous dissipation of energy only if it moves as a whole like a rigid body (Lamb, p. 549). Therefore if it is constrained to have zero velocity over a fixed finite area of surface it cannot move at all without dissipating energy into heat.

2. The energy equation for a fluid confined within a fixed boundary† at which it has zero velocity is

\[
\frac{dK}{dt} = -f + \int \int \int V \cdot X \, dt
\]

(1)

where \( V = (u,v,w) \) is the velocity of the fluid; \( X = (X,Y,Z) \) is the field force acting on unit volume of fluid; \( dt \) denotes the element of volume, the integration being throughout the entire region in question; \( K \) is the total kinetic energy of the fluid in the region; and \( f \) is the dissipation. The quantity \( f \) is inherently positive, and can vanish only if the fluid is at rest throughout the region. Equation (1) is perhaps sufficiently obvious, but it may be derived by multiplying the

* More exactly, a connected mass of such a fluid. Two separate portions of fluid can, of course, move relatively to each other without dissipation.

† Here, as throughout, the boundary may consist of one or more internal closed surfaces in addition to the external one; i.e. the region it bounds may be periphrastic (Lamb, p. 38).
first three equations of (5) below by \( u, v, w \) respectively, and then adding, integrating, and transforming suitably (Lamb, p. 8).

If the volume forces have at each instant a potential, \( X(t) = -\nabla \Omega(t) \), the last term drops out* leaving simply

\[
\frac{dK}{dt} = -f. \tag{2}
\]

Thus under the operation of potential forces there is a unique steady state, namely that wherein the fluid is at rest throughout the region. This is furthermore a stable state, since any imported motion dies out to zero by dissipating its kinetic energy into heat. The steady state attained is independent of the force field \( X \), provided that at each instant it has a potential \( \Omega \). The dissipation in the steady motion is less than in any other motion having the same boundary velocities.

3. For a specified motion, \( K \) in (2) is proportional to the fluid density \( \rho \), while \( f \) is proportional to the viscosity coefficient \( \eta \). Thus it is seen that the rate at which the motion dies out increases with increasing \( \mu = \eta/\rho \). Some idea of this may be gained by supposing that the velocity components were to die out uniformly throughout the region, i.e. \( u = u_0 \gamma(t) \), etc. Then \( K \) and \( f \) both vary as \( \gamma^2 \), and upon integrating (2) we obtain

\[
\gamma = e^{-\alpha t}, \tag{3}
\]

where \( \alpha > 0 \) depends only upon the initial velocities and is homogeneous of degree zero in them. Thus \( \alpha \) plays somewhat the role of an exponential decay factor. It was termed by Maxwell the kinematical viscosity.

4. Next consider the case where non-zero velocities for the fluid are prescribed over the boundary of a moving region. These boundary velocities are not entirely arbitrary, since the fluid motion must at each instant satisfy the equation of continuity \( \nabla \cdot V = 0 \) throughout the region. This requires that the relation

\[
\int V \cdot ds = 0 \tag{4}
\]

be satisfied identically on the moving boundary. Apart from this restriction the boundary velocities of the fluid may be assigned arbitrarily as functions of time.

5. The complete equations of motion of an incompressible fluid are

* By transforming to a surface integral over the boundary,

\[
\int V \cdot X \, dr = -\int \Omega \cdot V \, ds,
\]

and remembering that \( V = 0 \) on the boundary.