STEIN'S POSITIVE PART ESTIMATOR AND BAYES ESTIMATOR

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Summary

Stein's positive part estimator for \( p \) normal means is known to dominate the M.L.E. if \( p \geq 3 \). In this article by introducing some priors we show that Stein's positive part estimator is posterior mode. We also consider the Bayes estimators (posterior mean) with respect to the same priors and show that some of them dominate M.L.E. and are admissible.

1. Introduction

Let \( X \) have the \( p \)-variate normal distribution with unknown mean vector \( \theta \) and covariance matrix \( I \), and let the loss be quadratic, given by

\[
L(\theta, \hat{\theta}) = \| \hat{\theta} - \theta \|^2,
\]

where \( \hat{\theta} \) is the vector of estimate. Stein [5] showed that the estimator \( \hat{\theta}(X) = X \) is inadmissible when \( p \geq 3 \). James and Stein [2] showed that the estimator

\[
\hat{\theta}(X) = \left( 1 - \frac{t(p-2)}{\|X\|^2} \right) X, \quad 0 < t < 2
\]

dominates \( X \) and the uniformly best value of \( t \) is the James-Stein choice \( t = 1 \). But Stein [6] showed that the estimator

\[
\hat{\theta}(X) = \left( 1 - \min \left( 1, \frac{t(p-2)}{\|X\|^2} \right) \right) X, \quad 0 < t < 2
\]

dominates the above estimator. This estimator is called Stein's positive part estimator. Efron and Morris [1] gave its justification by Empirical Bayes approach. In this article we show that Stein's positive part estimators are posterior mode with respect to properly selected priors on \( \theta \). We also consider Bayes estimates (posterior mean) with respect to the priors and show the condition under which they dominate \( X \) and
are admissible.

2. Posterior mode and mean

A generalized prior distribution \(\pi_0(\theta)\) of \(\theta\), conditional on \(\lambda\), is given by the density

\[
\pi_0(\theta) = \left[\frac{\lambda}{2\pi(1-\lambda)}\right]^{p/2} \exp \left\{-\frac{\lambda}{2(1-\lambda)} \|\theta\|^2\right\}, \quad 0<\lambda<1,
\]

and \(\lambda\) has the density

\[
h(\lambda) \propto (1-\lambda)^{(a-1)/2}, \quad a=1-t(p-2)/p.
\]

Then it follows that the posterior density of \((\theta, \lambda)\) with respect to the generalized prior with the density

\[
\pi(\theta, \lambda) = \pi_0(\theta)h(\lambda),
\]

is

\[
p_x(\theta, \lambda) = \text{const.} \times \lambda^{(p-2)/2} \exp \left\{-\frac{1}{2} \left(\frac{\lambda}{1-\lambda} \|\theta-\lambda X\|^2 + \lambda \|X\|^2\right)\right\}.
\]

We have the following result.

**Theorem 1.** *Stein's positive part estimator is a posterior mode of \(\theta\) with respect to the above prior.*

**Proof.** From (2.4) we have

\[
2 \log p_x(\theta, \lambda) = \text{const.} + t(p-2) \log \lambda - \frac{\lambda}{1-\lambda} \|\theta-(1-\lambda)X\|^2 - \lambda \|X\|^2.
\]

If we denote the posterior mode of \((\theta, \lambda)\) by \((\theta^*, \lambda^*)\), then from (2.5) it follows that

\[
\theta^* = (1-\lambda^*)X
\]

and \(\lambda^*\) is the value which maximizes the following function

\[
g(\lambda) = t(p-2) \log \lambda - \lambda \|X\|^2.
\]

It is easily shown that \(\lambda^*\) is given by

\[
\lambda^* = \min \left(1, \frac{t(p-2)}{\|X\|^2}\right).
\]

Then from (2.6) and (2.8) we have the conclusion.

From (2.4) Bayes estimator with respect to the above prior is