INSTABILITY OF HAGEN—POISEUILLE FLOW FOR AXISYMMETRIC MODE

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Abstract

An investigation is described for instability problem of flow through a pipe of circular cross section. As a disturbance motion, we consider an axisymmetric nonlinear mode. An associated amplitude or modulation equation has been derived for this perturbation. This equation belongs to the diffusion type. The coefficient of it can be negative with Reynolds number increasing, because of the complex interaction between molecular diffusion and convection. The negative diffusion, when it occurs, causes a concentration and focusing of energy within the decaying slug, acting as a role of reversing natural decays.

I. Introduction

The instability problem of Hagen-Poiseuille flow has been studied, analytically, numerically, and experimentally, by many workers who research on hydrodynamic stability. Despite the growth in understanding the evolutionary process of disturbances, the original stability problem for this flow has remained a worry and a mystery. As with Plane Couette flow it is generally accepted that this basic flow is almost certainly stable to infinitesimal disturbance, according to linearized perturbation theory. But if the flow is kept very undisturbed, the flow in experiments can be kept laminar for value of Reynolds number up to 50,000 (Ekman, Taylor); otherwise, if finite-small amplitude disturbance are allowed turbulence can be occur for R as low as about 2,000.

Given that view, it follows that the flow is unstable only if nonlinear amplitude-depended forces can exert a significant enough influence on the behaviours of a small-but finite disturbance, hence a theoretical study of nonlinear stability properties is called for. There are several ways in which the attempts have been made to pinpiont an instability process in this flow.

a) Tatsum (1952) showed that the basic developing flow in the entry length of a pipe is unstable to linearized perturbations at or above a Reynolds number of order 10,000. This may be relevant for some experiments but, if the instability wave to persist into the fully developed flow region some nonlinear or other influence would be needed.

b) The weakly nonl near theory, suggested by Stuart (1958), (1960), (1978) and Watson (1960) while explaining many important physical phenomena in slightly subcritical or supercritical motions, alters the linear stability properties of a flow field only slightly by a relative amount of $O(1)$. But it is so strictly valid near a classical linear neutral curve. Since it is taken that no such neutral curve exists for hagen-Poiseuille flow, it is out of question in deciding the nonlinear stability of that...
flow. In particular, Davey and Nguyen (1979), Iton (1977) applied it on that flow, of course, there is no existence of a threshold amplitude establishment.

c) One important feature, which has been the subject of attention from Coles (1962), Wygnanski and Champagne (1971) and others, is the turbulent slug, which moves at some speed of propagation and has at front and rear rather sharp interfaces separating contorted turbulent vorticity within from “calm” or “laminar” flow outside. No satisfactory theoretical explanation of such slugs has been given.

Following Stuart’s theory (1981)—a theoretical basis for slowly varying slugs of long wavelength in case only perturbation to Hagen-Poiseuille flow is a swirl about axis. We studied a possible instability of that flow with general axisymmetrical disturbance, and showed that a property of negative diffusion arises because of the complex interaction between molecular diffusion and convection. The negative diffusion, when it occurs, causes a concentration and focusing of energy within the decaying slug and, reverse the natural decay.

II. Governing Equation of the Motion

We consider a flow under pressure through a long pipe of radius \( a \) and length \( L \). In undisturbed laminar flow, a uniform pressure gradient produces a velocity distribution which has a maximum value \( u_0 \) along the centre of the pipe. We use the cylindrical polar coordinates \((z, r, \theta)\), where \( r \) denotes the radius, \( \theta \), the angle and \( z \), the coordinate parallel to the pipe, the corresponding velocity components are \( u, \theta, w \) and we denote by \( p \) the pressure by \( T \), the time. To make our quantities dimensionless, we choose \( u_0, a, a^2/v \) and \( u_0^2 \) to be reference quantities for velocity, length, time and pressure, respectively. We denote by \( \phi \) the stream function. then Navier-Stokes equation for axisymmetrical motion in dimensionless form is given by:

\[
\frac{1}{R} \frac{\partial}{\partial T} \psi + \frac{1}{r} \frac{\partial (\phi, \psi)}{\partial (r, z)} + \frac{2}{r^2} \frac{\partial \phi}{\partial z} \psi = \frac{1}{R} \nabla^2 \psi
\]  
(2.1)

where

\[
\psi = \nabla^2 \phi = \left( \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \phi
\]  
(2.2)

and

\[
R = \frac{u_0 a}{v}, \quad u = \frac{1}{r} \frac{\partial \phi}{\partial r}, \quad w = -\frac{1}{r} \frac{\partial \phi}{\partial z}
\]  
(2.3)

\( R \) is the Reynolds number, and \( v \) the kinematic viscosity. \( \phi \) is bounded and satisfies

\[
\phi = \frac{\partial \phi}{\partial r} = 0 \quad \text{at} \quad r = 1
\]
(2.4)

\[
\phi_z \quad \text{regular} \quad \text{at} \quad r = 0
\]

Undisturbed basic flow for Hagen-Poiseuille motion is

\[
\phi_B = \frac{r^2 - r^4}{2 - 4}
\]  
(2.5)

We are interested in studying the behaviour of fluid motion under the influence of disturbance. In other words, we are interested in the evolution process of disturbance. For this purpose we let