Complementation in the lattice of subalgebras of a Boolean algebra

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Introduction

In this paper, we shall study the lattice of subalgebras of a Boolean algebra \( \mathcal{B} \). If \( S \) is a subset of \( \mathcal{B} \), then \((S)^*\) will denote the subalgebra of \( \mathcal{B} \) generated by \( S \). If \( B \) and \( C \) are subalgebras of \( \mathcal{B} \), we let \( B + C \) denote \((B \cup C)^*\). The collection of subalgebras of \( \mathcal{B} \) forms a lattice, \( \mathcal{L}(\mathcal{B}) \), under the operations of intersection and sum. We let \( 0_{\mathcal{B}} \), \( 1_{\mathcal{B}} \) denote the zero and one of \( \mathcal{B} \) and where there is no confusion we shall just write \( 0 \) and \( 1 \) instead of \( 0_{\mathcal{B}} \) and \( 1_{\mathcal{B}} \).

We shall explore the question of whether there is a reasonable notion of complement in \( \mathcal{L}(\mathcal{B}) \). We can look to the cases of the lattice of subsets of a given set or to the lattice of subspaces of a given vector space for analogies. If \( S \) is a set and \( A \subseteq S \), then the complement \( A \) is the largest set \( B \subseteq S \) such that \( B \cap A = \emptyset \). If \( V \) is a vector space and \( V_1 \) is a subspace of \( V \), then \( V_2 \) is a complementary subspace of \( V_1 \) and \( V_2 \) is a maximal element in the class of all subspaces \( W \) of \( V \) such that \( V_1 \cap W = \{0\} \) where \( 0 \) is the zero vector of \( V \). Thus as a first attempt, we say that if \( C \in \mathcal{L}(\mathcal{B}) \), then \( B \in \mathcal{L}(\mathcal{B}) \) is a complement of \( C \) if \( B \) is a maximal element in the class of \( A \in \mathcal{L}(\mathcal{B}) \) such that \( A \cap C = \{0, 1\} \). Or equivalently,

**DEFINITION 1.** If \( B \) and \( C \) are subalgebras of a Boolean algebra \( \mathcal{B} \), then \( B \) is a complement of \( C \) if \( B \cap C = \{0, 1\} \) and for any \( x \in \mathcal{B} - B \), \((\{x\} \cup B)^* \cap C = \{0, 1\} \).

Given a subalgebra \( C \in \mathcal{L}(\mathcal{B}) \), a simple application of Zorn’s lemma shows that there is a \( B \in \mathcal{L}(\mathcal{B}) \) such that \( B \) is a complement of \( C \). As is the case with vector spaces, in general a subalgebra \( C \) will not have a unique complement. The main question about our notion of complement is whether or not it is symmetric, that is, does \( B \) being a complement of \( C \) imply that \( C \) is a complement of \( B \)? In Section 1, we shall prove the answer is yes if \( \mathcal{B} \) is the Boolean algebra of finite and cofinite subsets of some set \( S \). In Section 2, we shall show that there are

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counterexamples in all other Boolean algebras \( \mathcal{B} \), that is, there are subalgebras \( B \) and \( C \) of \( \mathcal{B} \) such that \( B \) is a complement of \( C \) but \( C \) is not a complement of \( B \). Because of the results of Section 2, the following is a stronger notion than complement.

**DEFINITION.** If \( B \) and \( C \) are subalgebras of \( \mathcal{B} \), then \( B \) is a bicomplement of \( C \) if \( B \) is a complement of \( C \) and \( C \) is a complement of \( B \).

Again it is an easy application of Zorn's lemma to see that in any Boolean algebra, there are nontrivial pairs of subalgebra which are bicomplementary. For if \( A \) is a nontrivial subalgebra of \( \mathcal{B} \), then we let \( B \) be a complement of \( A \) and apply Zorn's lemma to find \( C \supseteq A \) such that \( C \) is a complement of \( B \). Clearly, \( B \) is a bicomplement of \( C \). However given the results of Section 2, it is not a priori clear that for every \( C \in \mathcal{L}(\mathcal{B}) \), there is a \( B \in \mathcal{L}(\mathcal{B}) \) such that \( B \) is a bicomplement of \( C \). In Section 3, we will prove that if \( \mathcal{B} \) is a directed union of countably many Boolean algebras which are Boolean algebras of finite and cofinite subsets of some set \( S \), then every subalgebra of \( \mathcal{B} \) has a bicomplement. In particular, this class of Boolean algebras include all the countable Boolean algebras.

1. **Preliminaries**

Given a Boolean algebra \( \mathcal{B} \), we know by the Stone Representation Theorem that \( \mathcal{B} \) is isomorphic to a field of subsets of a given set. Thus we will implicitly assume we are always dealing with a field of subsets of a given set. We shall let \( \vee, \wedge, \neg, \subseteq \) denote the meet, join, complement, and order relation of the Boolean algebra \( \mathcal{B} \).

Given \( a \in \mathcal{B} \) and \( B \) a subalgebra of \( \mathcal{B} \), one can easily show that any element of \( \langle \{a\} \cup B \rangle^* \) can be written in the form \( (a \land b_1) \lor (\neg a \land b_2) \) where \( b_1, b_2 \in B \). We shall use this fact repeatedly.

\( a \in \mathcal{B} \) is an atom of \( \mathcal{B} \) if \( a \neq 0 \) and for any \( b \in \mathcal{B} \), \( 0 \leq b \leq a \) implies either \( b = 0 \) or \( b = a \). \( a \in \mathcal{B} \) is atomless if \( a \neq 0 \) and there is no atom \( b \in \mathcal{B} \) such that \( b \leq a \). A Boolean algebra \( \mathcal{B} \) is atomless if \( \mathcal{B} \) contains no atoms. A Boolean algebra \( \mathcal{B} \) is atomic if \( \mathcal{B} \) contains no atomless elements. We let \( \mathcal{A}(\mathcal{B}) \) denote the set of atoms of \( \mathcal{B} \). If \( B \) is a subalgebra of \( \mathcal{B} \), \( \mathcal{A}(B) \) will denote the atoms of \( B \) relative to \( B \), i.e., \( x \in \mathcal{A}(B) \) is an atom in \( B \), but is not necessarily an atom in \( \mathcal{B} \). For any set \( S \), let \( \mathcal{I}(S) \) denote the ideal generated by \( S \). If \( \mathcal{B} \) is an atomic Boolean algebra, the derivative of \( \mathcal{B} \) is the quotient algebra \( \mathcal{B} \mod \mathcal{I}(\mathcal{A}(\mathcal{B})) \). It is a well known theorem of Vaught's [2], that if \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are countable atomic Boolean algebras, and \( \mathcal{B}_1 \mod \mathcal{I}(\mathcal{A}(\mathcal{B}_1)) \) is isomorphic to \( \mathcal{B}_2 \mod \mathcal{I}(\mathcal{A}(\mathcal{B}_2)) \) then \( \mathcal{B}_1 \) is