ON DAMPING COEFFICIENT DUE TO PHASE TRANSFORMATION

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ABSTRACT: The damping coefficient of capillary waves due to the evaporation-condensation process at the interface of the two phases of a fluid is evaluated. To highlight the mechanism of the effect of heat and mass transfer across the interface between regions of liquid and vapor, potential flow of incompressible fluids are assumed. Thus other mechanisms of damping are neglected. To facilitate the analysis, the method of multiple-scale is employed in the analysis, even though the problem is linear.

KEY WORDS: phase transformation, damping coefficient, capillary wave

1 INTRODUCTION

When an interface exists between two phases of a substance, such as water-vapor interface, there are invariably processes of phase transformation taking place. The effect of phase transformation on a dynamic process is normally very small. However, there are circumstances, for instance, when we are dealing with situations near the boiling point, that this effect may not be insignificant\(^1\). Therefore it may be of interest to have a definite idea of what and how various factors are combined to show this effect.

Evaporation and condensation normally are conspicuous when there is ambient heat transfer across the interface. For those cases, the thermal dissipation is easily recognizable, and the effect of phase transformation is to modify the dissipation due to heat transfer. Our purpose is to concentrate on the mechanism of phase transformation. So the problem we choose to study does not involve the ambient heat transfer across the interface. Rather, the process of transformation will cause a local temperature variation at the interface and thus induce a tiny flow of heat. The effect could be measurable for experimental studies in cryogenic fluids.

We have chosen a simple problem to analyze this process, i.e., the damping of capillary waves by the evaporation-condensation phase transformation at the interface. The analysis turns out not that straightforward, and we need to employ the method of multiple-scale to compute the damping coefficient of this linear problem. In the following, we shall first formulate the problem, and then carry out the analysis.

2 FORMULATION OF THE PROBLEM

We shall consider a two-dimensional liquid-vapor system with the dividing line of the liquid and vapor given by

\[
S(x,t) = y - \eta(x,t) = 0 \tag{1}
\]

The liquid is taken to be an incompressible inviscid fluid. We shall assume the vapor is incompressible and inviscid so far as dynamics is concerned. Thermodynamically, however, the pressure of vapor on the liquid-vapor interface is to be determined by the equilibrium vapor pressure \(p_v(T)\), where \(T\) is the temperature on the interface.

Let the liquid occupy the region \(-\infty < y < \eta(x,t)\). Since it is incompressible and inviscid, we shall just consider the potential flows. The continuity equation and the Bernoulli’s equation are thus

\[
\nabla^2 \phi = 0 \quad -\infty < y < \eta(x,t) \tag{2}
\]

and

\[
\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 = \text{constant} \quad -\infty < y < \eta(x,t) \tag{3}
\]

where \(\phi\) is the velocity potential, \(p\), the pressure, and \(\rho\), the density of the liquid.
The temperature field $T$ of the liquid is governed by the heat equation

$$\frac{\partial T}{\partial t} + (\nabla \phi) \cdot (\nabla T) = D \nabla^2 T$$

$$-\infty < y < \eta(x,t)$$

where $D$ is the thermal diffusion coefficient of the liquid.

The vapor occupies the region $\eta(x,t) < y < \infty$. Likewise, the governing equations are

$$\nabla^2 \phi' = 0$$

$$\eta(x,t) < y < \infty$$

$$\frac{\rho'}{\rho} + \frac{\partial \phi'}{\partial t} + \frac{1}{2}(\nabla \phi')^2 = \text{constant}$$

$$\eta(x,t) < y < \infty$$

and

$$\frac{\partial T'}{\partial t} + (\nabla \phi') \cdot (\nabla T') = D' \nabla^2 T'$$

$$\eta(x,t) < y < \infty$$

where $\phi'$ is the velocity potential, $\rho'$, the pressure, $\rho'$, the density, $T'$, the temperature, and $D'$, the thermal diffusion coefficient of the vapor.

The normal vector to the interface $S(x,t) = 0$ is $n = \nabla S / |\nabla S|$, and the velocity of the interface is $w = -[(\partial S / \partial t) \nabla S] / |\nabla S|^2$. The interfacial conditions on $S(x,t) = 0$ are then $[1,2]

$$\rho(v_n - w) = \rho'(v'_n - w)$$

$$\rho v_n \cdot (v_n - w) + p = \rho v'_n \cdot (v'_n - w) +$$

$$p' + \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

$$T = T'$$

$$p' = p_v(T')$$

and

$$\rho(v_n - w) \left[ \frac{1}{2}(v_n - w)^2 - \frac{1}{2}(v'_n - w)^2 - L \right] -$$

$$\kappa \left( \frac{\partial T}{\partial n} \right) = -\kappa' \left( \frac{\partial T'}{\partial n} \right)$$

where $v_n$ and $v'_n$ are the normal velocities, $\kappa$ and $\kappa'$ the thermal conduction coefficients, of the liquid and vapor, respectively, and $L$ is the latent heat of evaporation.

We are only interested in the damping of the small amplitude waves. The unperturbed state is a liquid-vapor system at rest divided by the flat interface: $y = 0$, and $\eta(x,t)$ is small. To be specific, the unperturbed equilibrium state is given by

$$S = y \quad v = v' = 0 \quad T = T' = T_0$$

$$p = p' = p_v(T_0) = p_0 \quad \rho = \rho_0 \quad \rho' = \rho'_0 = \frac{p_v(T_0)}{RT_0}$$

where $R$ is the universal gas constant, and we have assumed that the vapor obeys thermodynamically the perfect gas law.

To simplify the writing for the linearized problem, let us denote the deviations from the equilibrium values of the variables with the same symbols. The linearized version of the Eqs.(2)~(12) is then

$$\nabla^2 \phi = 0 \quad -\infty < y < 0$$

$$\frac{\rho}{\rho} + \frac{\partial \phi}{\partial t} = 0 \quad -\infty < y < 0$$

$$\frac{\partial T}{\partial t} + D \nabla^2 T = 0 \quad -\infty < y < 0$$

$$\nabla^2 \phi' = 0 \quad 0 < y < \infty$$

$$\frac{\rho'}{\rho} + \frac{\partial \phi'}{\partial t} = 0 \quad 0 < y < \infty$$

$$\frac{\partial T'}{\partial t} = D' \nabla^2 T' \quad 0 < y < \infty$$

and on $y = 0$

$$\rho \left( \frac{\partial \phi}{\partial y} - \frac{\partial \eta}{\partial t} \right) = \rho' \left( \frac{\partial \phi'}{\partial y} - \frac{\partial \eta}{\partial t} \right)$$

$$p = p' = \sigma \left( \frac{\partial^2 \eta}{\partial x^2} \right)$$

$$T = T'$$

$$p' = \beta T'$$

and

$$\rho \left( \frac{\partial \phi}{\partial y} - \frac{\partial \eta}{\partial t} \right) L + \kappa \left( \frac{\partial T}{\partial n} \right) = \kappa' \left( \frac{\partial T'}{\partial n} \right)$$

where we have denoted $\beta = (dp_v/dT)(T_0)$.

We note that because we are dealing with small amplitude waves, i.e., $\eta$ is small, the interface is approximately $y = 0$. Moreover, the normal velocities of the fluids at the interface are approximately $\partial \phi / \partial y$ and $\partial \phi' / \partial y$, and the velocity of the interface is $\partial \eta / \partial t$.

### 3 NORMAL MODES

Since the problem is linear, we may take Fourier transform with respect to $x$, or equivalently, consider only one Fourier mode with $e^{ikx}$ for the $x$-dependence.