A finite algebra $A$ with $SP(A)$ not elementary

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Given a finite algebra $A$ we have the variety generated by $A$, or $\{A\}^e$, and the quasi-variety generated by $A$, or $\{A\}^q$. The former is identical with $HSP(A)$, the latter with $SP(A)$. Baker proved that if $\{A\}^e$ is congruence-distributive, then it is finitely axiomatizable. (We assume that $A$ has a finite similarity type.) All proofs to date of Baker's theorem rely in part on a result of Jónsson: if $A$ is finite and $\{A\}^e$ congruence-distributive, then $\{A\}^q = \mathcal{B}^q$ where $\mathcal{B}$ is a finite set of finite algebras. Now if it were true that whenever $\mathcal{B}$ is a finite set of finite algebras, then $\mathcal{B}^q$ is finitely axiomatizable, it would put Baker's theorem in a radically new light. However, it is false, as we shall show.

A still open problem (due to Jónsson): Suppose that $\mathcal{B}$ is a finite set of finite algebras and $\mathcal{B}^e = \mathcal{B}^q$. Does it follow that $\mathcal{B}^e$ is finitely axiomatizable?

Our construction is as follows. Given any graph $G = \langle V, E \rangle$ with vertex-set $V$, edge-set $E$, let $A(G)$ be the algebra $\langle A(G), R, S, + \rangle$ with universe $A(G) = \{R, S\} \cup V \cup ((V \times V) \setminus E)$ (disjoint union)

and operations $R, S$ (constants, $R \neq S$), + (binary):

$$x + y = \begin{cases} (x, y) & \text{if } (x, y) \in (V \times V) \setminus E \\ S & \text{if } (x, y) \in E \\ R & \text{otherwise.} \end{cases}$$

Now let $G = \langle V, E \rangle$ be the graph pictured below:

![Figure 1](image_url)

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and for each integer \( n \geq 3 \), let \( C_n \) be the \( n \)-cycle

\[
\langle \{0, 1, \ldots, n-1\}, \{(i, j) : j \equiv i + 1 \pmod{n} \} \rangle.
\]

**THEOREM.** \( \{A(G)\}^q \) is not finitely axiomatizable.

If it were, then it could be defined by a single universal sentence, hence there would exist an \( n \) such that any \( B \) belongs to \( SP(A(G)) \) as soon as all \( n-1 \)-generated subalgebras of \( B \) are in \( SP(A(G)) \). But we have

**LEMMA.** For \( n \geq 3 \), \( A(C_n) \not\in SP(A(G)) \); each \( n-1 \)-generated subalgebra of \( A(C_n) \) belongs to \( SP(A(G)) \).

Notice that \( \{A(G)\}^F \) is finitely axiomatizable, for the following identities serve:

\[
R + x \equiv x + R \equiv R, \quad x + (y + z) \equiv R
\]

\[
S + x \equiv x + S \equiv R, \quad (x + y) + z \equiv R.
\]

Before proving the lemma, we observe a corollary that may be of interest to the model-theorists.

**COROLLARY.** There exists a complete \( \kappa \)-categorical theory \( T \) of finite type such that the universal sentences valid in \( T \) are not finitely axiomatizable.

Let \( T \) be the theory of the Boolean power \( A[B] \) where \( A = A(G) \) and \( B \) is the countable atomless Boolean algebra. It is known that \( T \) is complete and \( \kappa \)-categorical (see [2]). But the universal closure of the model class of \( T \) is simply \( SP(A(G)) \).

**Proof of the lemma.** That \( A(C_n) \not\in SP(A(G)) \) is easy. If \( h \) maps the first algebra homomorphically into the second, then \( h(S) = S \); hence if \( (x, y) \) is an edge in \( C_n \) then \( x + y = S \) and \( h(x) + h(y) = S \), so \( (h(x), h(y)) \) is an edge in \( G \), and moreover \( h(x), h(y) \in V \). It follows that \( h \upharpoonright n \) maps \( C_n \) homomorphically into \( G \). Looking at Figure 1, we easily convince ourselves that \( h(0) \neq h(2) \) must hold. Thus no homomorphism from \( A(C_n) \) into \( A(G) \) separates \( 0, 2 \).

Upon discarding one point of \( C_n \) from \( A(C_n) \) one obtains a subalgebra of \( A(C_n) \). All these subalgebras are isomorphic one to another, and every \( n-1 \)-generated subalgebra of \( A(C_n) \) is included in one of them. So to conclude the proof, we take for \( B \) the subalgebra of \( A(C_n) \) whose universe is \( B = A(C_n) \setminus \{0\} \) and show that the homomorphisms of \( B \) into \( A(G) \) separate the points of \( B \).